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Exponential Decay for Soft Potentials near Maxwellian

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Abstract

We consider both soft potentials with angular cutoff and Landau collision kernels in the Boltzmann theory inside a periodic box. We prove that any smooth perturbation near a given Maxwellian approaches zero at the rate of $e^{-\lambda t^p}$ for some $\lambda > 0$ and $0 < p < 1$. Our method is based on an unified energy estimate with appropriate exponential velocity weight. Our results extend the classical result Caflisch of [2] to the case of very soft potential and Coulomb interactions, and also improve the recent "almost exponential" decay results by [5, 14].

1. Introduction

In this article, we are concerned with soft potentials and Landau collision kernels in the Boltzmann theory for dynamics of dilute particles in a periodic box. Recall the Boltzmann equation as

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \quad F(0, x, v) = F_0(x, v), \tag{1}$$

where $F(t, x, v)$ is the spatially periodic distribution function for the particles at time $t \geqq 0$, position $x = (x_1, x_2, x_3) \in \mathbb{T}^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The left-hand side of this equation models the transport of particles and the operator on the right-hand side models the effect of collisions on the transport

$$Q(F, G) \equiv \int_{\mathbb{R}^3 \times S^2} |u - v|^\gamma B(\theta) \{F(u')G(v') - F(u)G(v)\} du d\omega.$$

Here $F(u) = F(t, x, u)$ etc. The exponent is $\gamma = 1 - \frac{4}{s}$ with $1 < s < 4$; we assume

$$-3 < \gamma < 0,$$

(soft potentials) and $B(\theta)$ satisfies the Grad angular cutoff assumption

$$0 < B(\theta) \leqq C|\cos\theta|. \tag{2}$$

Moreover, the post-collisional velocities satisfy

$$v' = v + [(u-v)\cdot\omega]\omega, \qquad u' = u - [(u-v)\cdot\omega]\omega, \tag{3}$$

$$|u|^2 + |v|^2 = |u'|^2 + |v'|^2 \tag{4}$$

and θ is defined by $\cos\theta = [u-v]\cdot\omega/|u-v|$.

On the other hand, the Landau equation is formally obtained in a singular limit of the Boltzmann equation. It can also be written as (1) but the collision operator is given by

$$\begin{aligned} Q(F,G) &= \nabla_v \cdot \left\{ \int_{\mathbb{R}^3} \phi(v-u)[F(u)\nabla_v G(v) - G(v)\nabla_u F(u)]du \right\} \\ &= \sum_{i,j=1}^{3} \partial_i \int_{\mathbb{R}^3} \phi^{ij}(v-u)[F(u)\partial_j G(v) - G(v)\partial_j F(u)]du, \end{aligned}$$

where $\partial_i = \partial_{v_i}$ etc. The non-negative matrix ϕ is given by

$$\phi^{ij}(v) = \left\{ \delta_{ij} - \frac{v_i v_j}{|v|^2} \right\} |v|^{2+\gamma}.$$

We assume soft potentials, which means $-3 \leqq \gamma < -2$ in this case. The original Landau collision operator with Coulombic interactions corresponds to $\gamma = -3$.

Denote the steady state Maxwellian by

$$\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}.$$

We perturb around the Maxwellian as

$$F(t,x,v) = \mu(v) + \sqrt{\mu(v)} f(t,x,v).$$

Then the initial value problem (1) can be rewritten as

$$[\partial_t + v\cdot\nabla_x]f + Lf = \Gamma[f,f], \;\; f(0,x,v) = f_0(x,v), \tag{5}$$

where L is the linear part of the collision operator, Q, and Γ is the nonlinear part.

For the Boltzmann equation, the standard linear operator [6] is

$$Lg = \nu(v)g - Kg, \tag{6}$$

where the collision frequency is

$$\nu(v) = \int B(\theta)|v-u|^{\gamma}\mu(u)\,du\,d\omega. \tag{7}$$

The operators K and Γ, in the Boltzmann case, are defined in (19), (20) and (34).

For the Landau equation, the linear operator [9] is

$$Lg = -Ag - Kg, \tag{8}$$

with A, K and Γ defined in (47), (48) and (49). The Landau collision frequency is

$$\sigma^{ij}(v) = \phi^{ij} * \mu = \int_{\mathbb{R}^3} \phi^{ij}(v-u)\mu(u)du. \tag{9}$$

We remark that $\sigma^{ij}(v)$ is a positive self-adjoint matrix [4].

Notation: Let $\langle \cdot, \cdot \rangle$ denote the standard $L^2(\mathbb{R}^3)$ inner product. We also use $(\cdot, \cdot)$ to denote the standard $L^2(\mathbb{T}^3 \times \mathbb{R}^3)$ inner product with corresponding L^2 norm $\| \cdot \|$. Define a weight function in v by

$$w = w(\ell, \vartheta)(v) \equiv (1+|v|^2)^{\tau\ell/2} \exp\left(\frac{q}{4}(1+|v|^2)^{\frac{\vartheta}{2}}\right). \tag{10}$$

Here $\tau < 0$, $\ell \in \mathbb{R}$, $0 < q$ and $0 \leqq \vartheta \leqq 2$. Denote weighted L^2 norms as

$$|g|^2_{\ell,\vartheta} \equiv \int_{\mathbb{R}^3} w^2(\ell,\vartheta)|g|^2 dv, \quad ||g||^2_{\ell,\vartheta} \equiv \int_{\mathbb{T}^3} |g|^2_{\ell,\vartheta} dx.$$

For the Boltzmann equation, define the weighted dissipation norm as

$$|g|^2_{\boldsymbol{\nu},\ell,\vartheta} \equiv \int_{\mathbb{R}^3} w^2(\ell,\vartheta)\nu(v)|g(v)|^2 dv, \tag{11}$$
$$||g||^2_{\boldsymbol{\nu},\ell,\vartheta} \equiv \int_{\mathbb{T}^3} |g|^2_{\boldsymbol{\nu},\ell,\vartheta} dx.$$

For the Landau equation, define the weighted dissipation norm as

$$|g|^2_{\boldsymbol{\sigma},\ell,\vartheta} \equiv \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} w^2(\ell,\vartheta)\left\{\sigma^{ij}\partial_i g \partial_j g + \sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}|g|^2\right\} dv, \tag{12}$$
$$||g||^2_{\boldsymbol{\sigma},\ell,\vartheta} \equiv \int_{\mathbb{T}^3} |g|^2_{\boldsymbol{\sigma},\ell,\vartheta} dx.$$

Since our proof of decay does not depend upon detailed properties which are specific to either dissipation norm, sometimes we unify the notation as $||g||_{\boldsymbol{D},\ell,\vartheta}$, which denotes either $||g||_{\boldsymbol{\nu},\ell,\vartheta}$ or $||g||_{\boldsymbol{\sigma},\ell,\vartheta}$. If $\vartheta = 0$ then we drop the index, for example $||g||_{\boldsymbol{D},\ell,0} = ||g||_{\boldsymbol{D},\ell}$ and the same for the other norms.

Next define a high order derivative

$$\partial^\alpha_\beta \equiv \partial^{\alpha^0}_t \partial^{\alpha^1}_{x_1} \partial^{\alpha^2}_{x_2} \partial^{\alpha^3}_{x_3} \partial^{\beta^1}_{v_1} \partial^{\beta^2}_{v_2} \partial^{\beta^3}_{v_3},$$

where $\alpha = [\alpha^0, \alpha^1, \alpha^2, \alpha^3]$ is the multi-index related to the space-time derivative and $\beta = [\beta^1, \beta^2, \beta^3]$ is the multi-index related to the velocity derivatives. If each component of β is not greater than that of β_1's, we denote by $\beta \leqq \beta_1$; $\beta < \beta_1$ means $\beta \leqq \beta_1$ and $|\beta| < |\beta_1|$. We also denote $\binom{\beta}{\beta_1}$ by $C^{\beta_1}_\beta$.

Fix $N \geqq 8$ and $l \geqq 0$. An "**instant energy functional**" satisfies

$$\frac{1}{C}\mathcal{E}_{l,\vartheta}(g)(t) \leqq \sum_{|\alpha|+|\beta|\leqq N} ||\partial^{\alpha}_{\beta} g(t)||^2_{|\beta|-l,\vartheta} \leqq C\mathcal{E}_{l,\vartheta}(g)(t). \tag{13}$$

If g is independent of $t \geqq 0$, then the temporal derivatives are defined through equation (5). Further, the "**dissipation rate**" is given by

$$\mathcal{D}_{l,\vartheta}(g)(t) \equiv \sum_{|\alpha|+|\beta|\leqq N} ||\partial^{\alpha}_{\beta} g(t)||^2_{\boldsymbol{D},|\beta|-l,\vartheta}. \tag{14}$$

We will also write $\mathcal{E}_{l,0}(g)(t) = \mathcal{E}_l(g)(t)$ and $\mathcal{D}_{l,0}(g)(t) = \mathcal{D}_l(g)(t)$. We note from (10) that for $l > 0$ these norms contain a polynomial factor $(1+|v|^2)^{-\tau l/2}$. The weight factor $(1+|v|^2)^{\tau|\beta|/2}$ (dependent on the number of velocity derivatives) is designed to control the streaming term $v \cdot \nabla_x f$.

If initially $F_0(x,v) = \mu(v) + \sqrt{\mu(v)} f_0(x,v)$ has the same mass, momentum and total energy as the Maxwellian μ, then formally for any $t \geqq 0$ we have

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f(t)\mu^{1/2} = \int_{\mathbb{T}^3\times\mathbb{R}^3} v_i f(t)\mu^{1/2} = \int_{\mathbb{T}^3\times\mathbb{R}^3} |v|^2 f(t)\mu^{1/2} = 0. \tag{15}$$

We are now ready to state the main result.

Theorem 1. *Let* $N \geqq 8$*,* $l \geqq 0$*,* $0 \leqq \vartheta \leqq 2$ *and* $0 < q$*. If* $\vartheta = 2$*, then further assume* $q < 1$*. Choose initial data* $F_0(x,v) = \mu(v) + \sqrt{\mu(v)} f_0(x,v)$ *such that* $f_0(x,v)$ *satisfies* (15)*. In* (10)*, for the Boltzmann case assume* $\tau \leqq \gamma$ *and for the Landau case assume* $\tau \leqq -1$*.*

Then there exists an instant energy functional $\mathcal{E}_{l,\vartheta}(f)(t)$ *such that if* $\mathcal{E}_{l,\vartheta}(f_0)$ *is sufficiently small, then the unique global solution to* (1) *in both the Boltzmann case and the Landau case satisfies*

$$\frac{d}{dt}\mathcal{E}_{l,\vartheta}(f)(t) + \mathcal{D}_{l,\vartheta}(f)(t) \leqq 0. \tag{16}$$

In particular,

$$\sup_{0\leqq s\leqq\infty} \mathcal{E}_{l,\vartheta}(f)(s) \leqq \mathcal{E}_{l,\vartheta}(f_0). \tag{17}$$

Moreover, if $\vartheta > 0$*, then there exists* $\lambda > 0$ *such that*

$$\mathcal{E}_l(f)(t) \leqq Ce^{-\lambda t^p}\mathcal{E}_{l,\vartheta}(f_0),$$

where in the Boltzmann case

$$p = p(\vartheta,\gamma) = \frac{\vartheta}{\vartheta - \gamma},$$

and in the Landau case

$$p = p(\vartheta,\gamma) = \frac{\vartheta}{\vartheta - (2+\gamma)}.$$

In the second authors papers [9, 10], (16) was established with $\tau = \gamma$ in the Boltzmann case, $\tau = 2+\gamma$ in the Landau case and $\vartheta = l = 0$. We extended (16) to the case $l \geqq 0$ in [14]. There we used (16) and (17) to establish the following theorem via direct interpolation for $\vartheta = 0$.

Theorem 2. *Assume everything from Theorem 1 and fix $k > 0$. In addition, if $\mathcal{E}_{l+k,\vartheta}(f_0)$ is sufficiently small then*

$$\mathcal{E}_{l,\vartheta}(f)(t) \leqq C_{l,k}\left(1+\frac{t}{k}\right)^{-k}\mathcal{E}_{l+k,\vartheta}(f_0).$$

For $\vartheta > 0$, the proof of Theorem 2 is exactly the same as in [14]. Even though, in [14], Coulomb interactions ($\gamma = -3$) are assumed for the Landau case the proof of Theorem 2 works exactly the same for $\tau \leqq -1$ and $-3 \leqq \gamma < -2$. However we remark that for $\tau < 2+\gamma$ in the Laudau case, the interpolations used to prove Theorem 2 are not optimal.

The main difficulty in proving any kind of decay for soft potentials is caused by the lack of a spectral gap for both linear operators (6) and (8). In the Boltzmann case, the dominant part of the linear operator (6) is of the form

$$\frac{1}{C}(1+|v|^2)^{\gamma/2} \leqq \nu(v) \leqq C(1+|v|^2)^{\gamma/2}, \quad C>0, \text{ for } \gamma < 0. \tag{18}$$

From another point of view, at high velocities the dissipation is much weaker than the instant energy. However, Theorem 1 and Theorem 2 show that given explicit control over $f(t,x,v)$ at high velocities, no matter how weak, we can obtain a precise decay rate. On the other hand, it is difficult to construct solutions with a weight stronger than (10) with $\vartheta = 2$. From this point of view, Theorem 1 and Theorem 2 together form a rather satisfactory theory of convergence rates to Maxwellian for soft potentials and Landau operators, in a context close to equilibrium.

The constants in our estimates are certainly not optimal or explicit in all cases. However, $p = \frac{\vartheta}{\vartheta-\gamma}$ comes from the following simplification of the Boltzmann equation [1]:

$$\partial_t f(t,|v|) + |v|^\gamma f(t,|v|) = 0, \quad -3<\gamma<0, \quad |v|>0.$$

Consider initial data with rapid decay as required by our norms

$$f(0,|v|) = e^{-c|v|^\vartheta}, \quad c>0, \ \ 0<\vartheta \leqq 2.$$

Then the solution to this system is exactly

$$f(t,|v|) = e^{-c|v|^\vartheta - t|v|^\gamma}.$$

By splitting into $\{|v| \geqq t^{p/\vartheta}\}$ and $\{|v| < t^{p/\vartheta}\}$ one can show that

$$c_0 e^{-c_1 t^p} \leqq \int_{|v|>0} |f(t,|v|)|^2 d|v| \leqq c_2 e^{-c_3 t^p},$$

with $c_i > 0$ $(i = 0,1,2,3)$.

The study of trend to Maxwellians is important in kinetic theory both from physical and mathematical standpoints. In a periodic box, it was UKAI [17] who obtained exponential convergence (with $p = 1$), and hence constructed the first global in time solutions in the spatially inhomogeneous Boltzmann theory. He treated the case of a cutoff hard potential. In 1980, CAFLISCH [1, 2] established exponential decay (with the same $p(2, \gamma)$) as well as global in time solutions for the Boltzmann equation with potentials which are not too soft ($-1 < \gamma < 0$). About the same time, in the whole space setting, also for cutoff soft potentials with $\gamma > -1$, UKAI and GUO [18] obtained the rate $O(t^{-\alpha})$ with $0 < \alpha < 1$; their optimal case in $\mathbb{R}^3$ yields $\alpha = 3/4$. In these early investigations, a sufficiently fast time decay of the linearized Boltzmann equation around a Maxwellian played the crucial role in bootstrapping to the full nonlinear dynamics. For the soft potentials, such linear decay estimates can be very difficult and delicate. It has thus been an open problem to study the decay property as well as to construct global in time smooth solutions for very soft potentials with γ near -3.

Recently, a nonlinear energy method for constructing global solutions was developed by GUO to avoid using the linear decay. Indeed, by showing the linearized collision operator was always positive definite along the full nonlinear dynamics, global in time smooth solutions near Maxwellians were constructed for all cutoff soft potentials of $-3 < \gamma < 0$ [10], even for the Landau equation with Coulomb interaction [9]. However, the time decay of such solutions was left open. See [8, 11–13, 15, 16] for more applications of such a method.

From a completely different approach, DESVILLETTES and VILLANI [5] have recently developed a framework to study the trend to Maxwellians for *general* smooth solutions, not necessarily near any Maxwellian. As an application, their method leads to the almost exponential decay rates (i.e., faster than any given polynomial) for smooth solutions constructed earlier by the for all cutoff soft potentials and the Landau equation.

Inspired by such a striking result, in [14] we re-examined and improved the energy method to give a more direct proof of such almost exponential decay in the close to Maxwellian setting. We introduced a family of polynomial velocity weight functions and used some simple interpolation techniques. It is interesting to note that our decay estimate is a consequence of the weighted energy estimate for the global nonlinear solution, not the other way around as in earlier methods [1, 2, 17, 18]. It is clear from our analysis that a stronger velocity weight yields faster time decay. On the other hand, the time decay rate could be very slow without additional velocity weight, as seen in [3].

It is thus very natural to try to use exponential velocity weight functions to get exponential time decay, which is the main purpose of our current investigation. The key is to show that the new energy with an exponential velocity weight is bounded for all time. In order to carry out such an energy estimate, we follow the general framework and strategy in [10, 9, 14]. However, many new analytical difficulties arise and we have to develop new techniques accordingly. The main difficulty lies in the estimates for linearized collision operators around Maxwellians. The presence of the exponential weight factor $\exp\{\frac{q}{4}(1 + |v|^2)^{\vartheta/2}\}$ in (10) requires much more precise estimates at almost all levels. In the case of a cutoff soft potential, a careful application of the Caflisch estimate (Lemma 1) is combined with the splitting trick in [10] to treat the very soft potential of $-3 < \gamma \leqq -1$. Furthermore, to estimate the trickiest terms in Lemma 2, we found a version of energy conservation (4) for the variables (23) in the Hilbert–Schmidt form for K in (31). On the other hand, in the Landau case, an extra v factor from the derivative of the weight $\exp\{\frac{q}{4}(1+|v|^2)^{\vartheta/2}\}$ creates the most challenging difficulty to close the estimate in the same norm. We have to use different weight functions (that appeared in the norm (12)), very precisely to balance between the derivative part $\sigma^{ij}\partial_i g \partial_j g$ and the no derivative part $\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}|g|^2$ in Lemma 8. Thus $\tau \leqq -1$ is assumed. Moreover, in Lemma 9, we have to introduce a new splitting of the linear Landau operator in the $\vartheta = 2$ case where $q < 1$ is crucially used.

The paper is organized as follows. In Section 2, we establish the estimates with the exponential weight (10) for the Boltzmann equation. In Section 3, we establish estimates with weight (10) for the Landau equation. Finally, in Section 4 we establish the crucial energy estimate uniformly for both cases. Some details are exactly the same as in [9, 10, 14]. We will sketch these details which can be found elsewhere. Finally, we prove exponential decay in Section 5 using the global bound (17).

2. Boltzmann Estimates

In this section, we will prove the basic estimates used to obtain global existence of solutions with an exponential weight in the Boltzmann case. These estimates are similar to those in [10], but here the exponential weight which was not present earlier forces us to modify the proofs and some of the estimates. We will use the classical soft potential estimate of Caflisch [1] (Lemma 1) with v derivatives to estimate the linear operator (Lemma 2). We discuss the new features of each proof after the statement of each Lemma.

Recall K and Γ from (6) and (5). $K = K_2 - K_1$ is defined as [6, 7]:

$$[K_1 g](v) = \int_{\mathbb{R}^3\times S^2} B(\theta)|u-v|^\gamma \mu^{1/2}(u)\mu^{1/2}(v)g(u)dud\omega, \tag{19}$$

$$\begin{aligned}[K_2 g](v) &= \int_{\mathbb{R}^3\times S^2} B(\theta)|u-v|^\gamma \mu^{1/2}(u)\mu^{1/2}(u')g(v')dud\omega \\ &\quad + \int_{\mathbb{R}^3\times S^2} B(\theta)|u-v|^\gamma \mu^{1/2}(u)\mu^{1/2}(v')g(u')dud\omega.\end{aligned} \tag{20}$$

Consider a smooth cutoff function $0 \leqq \chi_m \leqq 1$ such that (for $m > 0$)

$$\chi_m(s) \equiv 1, \text{ for } s \geqq 2m; \quad \chi_m(s) \equiv 0, \text{ for } s \leqq m. \tag{21}$$

Then define $\bar{\chi}_m = 1 - \chi_m$. Use χ_m to split $K_2 = K_2^\chi + K_2^{1-\chi}$, where

$$\begin{aligned}K_2^\chi g \equiv &\int_{\mathbb{R}^3\times S^2} B(\theta)|u-v|^\gamma \mu^{1/2}(u)\chi_m(|u-v|)\mu^{1/2}(u')g(v')dud\omega \\ &+ \int_{\mathbb{R}^3\times S^2} B(\theta)|u-v|^\gamma \mu^{1/2}(u)\chi_m(|u-v|)\mu^{1/2}(v')g(u')dud\omega.\end{aligned}$$

After removing the singularity at $u = v$, following the procedure in [6, 7] (see also equations. (35) and (36) in [10]), we can write

$$K_2^{\chi} g = \int_{\mathbb{R}^3} k_2^{\chi}(v,\xi) g(v+\xi) d\xi,$$

where

$$k_2^{\chi}(v,\xi) \equiv \frac{e^{-\frac{1}{8}|\xi|^2 - \frac{1}{2}|\zeta_{\|}|^2}}{|\xi|\sqrt{\pi^3/2}} \int_{\mathbb{R}^2} \frac{\chi_m(\sqrt{|\xi|^2+|\xi_{\perp}|^2})}{\left(|\xi|^2+|\xi_{\perp}|^2\right)^{\frac{1-\gamma}{2}}} e^{-\frac{1}{2}|\xi_{\perp}+\zeta_{\perp}|^2} \frac{B(\theta)}{|\cos\theta|} d\xi_{\perp}. \tag{22}$$

The integration variables are $d\xi_{\perp} = d\xi_{\perp}^1 d\xi_{\perp}^2$ but $\xi_{\perp} = \xi_{\perp}^1 \xi^1 + \xi_{\perp}^2 \xi^2 \in \mathbb{R}^3$ where $\{\xi^1, \xi^2, \xi/|\xi|\}$ is an orthonormal basis for $\mathbb{R}^3$. Also

$$\zeta_{\|} = \frac{(v\cdot\xi)\xi}{|\xi|^2} + \frac{1}{2}\xi, \quad \zeta_{\perp} = v - \frac{(v\cdot\xi)\xi}{|\xi|^2} = (v\cdot\xi^1)\xi^1 + (v\cdot\xi^2)\xi^2. \tag{23}$$

This formulation is well suited for taking high order v-derivatives. CAFLISCH [1] proved Lemma 1 below with no derivatives. In contrast, we have already removed the singularity in K_2^{χ}. We extend the estimate from $-1 < \gamma < 0$ to the full range $-3 < \gamma < 0$. As in [10], we will see in Lemma 2 that the singular part of K_2, $K_2^{1-\chi}$, has stronger decay.

Lemma 1. *For any multi-index β and any $0 < s_1 < s_2 < 1$,*

$$\left|\partial_\beta k_2^{\chi}(v,\xi)\right| \leqq C \frac{\exp\left(-\frac{s_2}{8}|\xi|^2 - \frac{s_1}{2}|\zeta_{\|}|^2\}\right)}{|\xi|(1+|v|+|\xi+v|)^{1-\gamma}}.$$

Here $C > 0$ will depend on s_1, s_2 and β.

Proof. Fix $0 < s_1 < s_2 < 1$. If $|\beta| > 0$, from (22) and (23) we have

$$\partial_\beta k_2^{\chi}(v,\xi) = \frac{e^{-\frac{1}{8}|\xi|^2}}{|\xi|\sqrt{\pi^3/2}} \int_{\mathbb{R}^2} \frac{\chi_m(\sqrt{|\xi|^2+|\xi_{\perp}|^2})}{\left(|\xi|^2+|\xi_{\perp}|^2\right)^{\frac{1-\gamma}{2}}} \partial_\beta \left(e^{-\frac{1}{2}|\xi_{\perp}+\zeta_{\perp}|^2 - \frac{1}{2}|\zeta_{\|}|^2}\right) \times \frac{B(\theta)}{|\cos\theta|} d\xi_{\perp}.$$

Recalling (23) and by a simple induction, for any $0 < q' < 1$, we have

$$\left|\partial_\beta \left(e^{-\frac{1}{2}|\xi_{\perp}+\zeta_{\perp}|^2 - \frac{1}{2}|\zeta_{\|}|^2}\right)\right| \leqq C(|\beta|, q') e^{-\frac{q'}{2}|\xi_{\perp}+\zeta_{\perp}|^2 - \frac{q'}{2}|\zeta_{\|}|^2}.$$

Further restrict $q' > s_1$. For $|\beta| \geqq 0$, using the last display and (2) we have

$$\left|\partial_\beta k_2^{\chi}(v,\xi)\right| \leqq C \frac{e^{-\frac{1}{8}|\xi|^2}}{|\xi|} \int_{\mathbb{R}^2} \frac{\chi_m(\sqrt{|\xi|^2+|\xi_{\perp}|^2})}{\left(|\xi|^2+|\xi_{\perp}|^2\right)^{\frac{1-\gamma}{2}}} e^{-\frac{q'}{2}|\xi_{\perp}+\zeta_{\perp}|^2 - \frac{q'}{2}|\zeta_{\|}|^2} d\xi_{\perp}.$$

Notice that from (23), $\zeta_\perp$ is independent of the integration variable $\xi_\perp$. Change variables $\xi_\perp \to \xi_\perp - \zeta_\perp$ on the right-hand side to obtain

$$C\frac{e^{-\frac{1}{8}|\xi|^2-\frac{q'}{2}|\zeta_\|\|^2}}{|\xi|}\int_{\mathbb{R}^2}\frac{\chi_m(\sqrt{|\xi|^2+|\xi_\perp-\zeta_\perp|^2})}{\left(|\xi|^2+|\xi_\perp-\zeta_\perp|^2\right)^{\frac{1-\gamma}{2}}}e^{-\frac{q'}{2}|\xi_\perp|^2}d\xi_\perp.$$

Using (21), then, we have

$$\left|\partial_\beta k_2^\chi(v,\xi)\right| \leqq C(m)\frac{e^{-\frac{1}{8}|\xi|^2-\frac{q'}{2}|\zeta_\|\|^2}}{|\xi|}\int_{\mathbb{R}^2}\frac{e^{-\frac{q'}{2}|\xi_\perp|^2}d\xi_\perp}{\left(1+|\xi|^2+|\xi_\perp-\zeta_\perp|^2\right)^{\frac{1-\gamma}{2}}}. \tag{24}$$

In the rest of the proof, we will refine this estimate by further splitting the integration region.

Choose any $q''>0$ with $q''<s_1$ and then define $\tau_* = \sqrt{\frac{q'-s_1}{q'-q''}} < 1$. Split the integration region as follows

$$\{|\xi_\perp| > \tau_*|\zeta_\perp|\} \cup \{|\xi_\perp| \leqq \tau_*|\zeta_\perp|\}.$$

Further split the right-hand side of (24) into $k_2^{\chi,1}(v,\xi)+k_2^{\chi,2}(v,\xi)$, where $k_2^{\chi,1}(v,\xi)$ is restricted to the region $\{|\xi_\perp| > \tau_*|\zeta_\perp|\}$:

$$k_2^{\chi,1}(v,\xi) \equiv C(m)\frac{e^{-\frac{1}{8}|\xi|^2-\frac{q'}{2}|\zeta_\|\|^2}}{|\xi|}\int_{|\xi_\perp|>\tau_*|\zeta_\perp|}\frac{e^{-\frac{q'}{2}|\xi_\perp|^2}d\xi_\perp}{\left(1+|\xi|^2+|\xi_\perp-\zeta_\perp|^2\right)^{\frac{1-\gamma}{2}}}.$$

For $k_2^{\chi,1}(v,\xi)$ we will observe exponential decay. And for $k_2^{\chi,2}(v,\xi)$ we can extract from the denominator on the right-hand side of (24) the exact decay stated in Lemma 1.

First consider $k_2^{\chi,1}(v,\xi)$. Since $\{|\xi_\perp| > \tau_*|\zeta_\perp|\}$ and $q'-q''>0$ we have

$$\begin{aligned}\left|k_2^{\chi,1}(v,\xi)\right| &\leqq C\frac{e^{-\frac{1}{8}|\xi|^2-\frac{q'}{2}|\zeta_\|\|^2}}{|\xi|}\int_{\{|\xi_\perp|>\tau_*|\zeta_\perp|\}}\frac{e^{-\frac{q''}{2}|\xi_\perp|^2-\frac{q'-q''}{2}|\xi_\perp|^2}}{\left(1+|\xi|^2+|\xi_\perp-\zeta_\perp|^2\right)^{\frac{1-\gamma}{2}}}d\xi_\perp \\ &\leqq C\frac{e^{-\frac{1}{8}|\xi|^2-\frac{q'}{2}|\zeta_\|\|^2}}{|\xi|}\int_{\{|\xi_\perp|>\tau_*|\zeta_\perp|\}}\frac{e^{-\frac{q''}{2}|\xi_\perp|^2-\frac{q'-s_1}{2}|\zeta_\perp|^2}}{\left(1+|\xi|^2+|\xi_\perp-\zeta_\perp|^2\right)^{\frac{1-\gamma}{2}}}d\xi_\perp.\end{aligned}$$

By (23),

$$|\zeta_\||^2+|\zeta_\perp|^2 = |\zeta_\| + \zeta_\perp|^2 = |v+\xi/2|^2.$$

Splitting $q' = s_1 + (q'-s_1)$ we have

$$\begin{aligned}\left|k_2^{\chi,1}(v,\xi)\right| &\leqq \frac{C}{|\xi|}e^{-\frac{1}{8}|\xi|^2-\frac{s_1}{2}|\zeta_\||^2}\int_{\{|\xi_\perp|>\tau_*|\zeta_\perp|\}}\frac{e^{-\frac{q''}{2}|\xi_\perp|^2-\frac{q'-s_1}{2}(|\zeta_\perp|^2+|\zeta_\||^2)}}{\left(1+|\xi|^2+|\xi_\perp-\zeta_\perp|^2\right)^{\frac{1-\gamma}{2}}}d\xi_\perp \\ &\leqq \frac{C}{|\xi|}e^{-\frac{1}{8}|\xi|^2-\frac{s_1}{2}|\zeta_\||^2-\frac{q'-s_1}{2}|v+\xi/2|^2}.\end{aligned}$$

This will be more than enough decay. We expand

$$\begin{aligned}|v+\xi/2|^2 &= |v|^2+\frac{1}{4}|\xi|^2+v\cdot\xi=\frac{1}{4}|v+\xi|^2+\frac{3}{4}|v|^2+\frac{1}{2}v\cdot\xi\\ &\geqq \frac{1}{4}|v+\xi|^2+\frac{3}{4}|v|^2-\frac{1}{4}|v|^2-\frac{1}{4}|\xi|^2 \\ &= \frac{1}{4}|v+\xi|^2+\frac{1}{2}|v|^2-\frac{1}{4}|\xi|^2.\end{aligned} \tag{25}$$

Plug the last display into the one above it to obtain

$$\begin{aligned}\left|k_2^{\chi,1}(v,\xi)\right| &\leqq \frac{C}{|\xi|}e^{-\frac{1}{8}|\xi|^2-\frac{s_1}{2}|\zeta_{\|}|^2}e^{-\frac{q'-s_1}{8}|v+\xi|^2-\frac{q'-s_1}{4}|v|^2+\frac{q'-s_1}{8}|\xi|^2}\\ &= \frac{C}{|\xi|}e^{-\frac{s_1+1-q'}{8}|\xi|^2-\frac{s_1}{2}|\zeta_{\|}|^2}e^{-\frac{q'-s_1}{8}|v+\xi|^2-\frac{q'-s_1}{4}|v|^2}.\end{aligned}$$

Given s_2 with $s_1 < s_2 < 1$, we can always choose q', restricted by $s_1 < q' < 1$, such that $s_2 = s_1 + 1 - q'$. This completes the estimate for $k_2^{\chi,1}(v,\xi)$.

On $\{|\xi_\perp| \leqq \tau_*|\zeta_\perp|\}$, $|\zeta_\perp - \xi_\perp| \geqq |\zeta_\perp| - |\xi_\perp| \geqq (1-\tau_*)|\zeta_\perp|$ $(0 < \tau_* < 1)$. Hence (24) over this region is bounded as

$$\begin{aligned}\left|k_2^{\chi,2}(v,\xi)\right| &\leqq \frac{Ce^{-\frac{1}{8}|\xi|^2-\frac{q'}{2}|\zeta_{\|}|^2}}{|\xi|\left(1+|\xi|^2+(1-\tau_*)^2|\zeta_\perp|^2\right)^{\frac{1-\gamma}{2}}}\int_{\{|\xi_\perp|\leqq\tau_*|\zeta_\perp|\}} e^{-\frac{q'}{2}|\xi_\perp|^2}d\xi_\perp\\ &\leqq \frac{Ce^{-\frac{1}{8}|\xi|^2-\frac{s_1}{2}|\zeta_{\|}|^2}}{|\xi|\left(1+|\xi|^2+|\zeta_\perp|^2+|\zeta_{\|}|^2\right)^{\frac{1-\gamma}{2}}}\\ &= \frac{Ce^{-\frac{1}{8}|\xi|^2-\frac{s_1}{2}|\zeta_{\|}|^2}}{|\xi|\left(1+|\xi|^2+|v+\xi/2|^2\right)^{\frac{1-\gamma}{2}}},\end{aligned}$$

where we used $s_1 < q'$ to absorb some powers of $|\zeta_{\|}|$, going from the first line to the second. Now plug (25) into the last display to complete the estimate. □

Next we will prove the energy estimates for the linear operator (6).

Lemma 2. *Let* $|\beta| > 0$, $\ell \in \mathbb{R}$, $0 \leqq \vartheta \leqq 2$ *and* $0 < q$. *If* $\vartheta = 2$ *restrict* $0 < q < 1$. *Then for all* $\eta > 0 \exists C(\eta) > 0$ *such that*

$$\langle w^2(\ell,\vartheta)\partial_\beta[\nu g],\partial_\beta g\rangle \geqq \left|\partial_\beta g\right|^2_{\nu,\ell,\vartheta} - \eta\sum_{|\beta_1|\leqq|\beta|}\left|\partial_{\beta_1}g\right|^2_{\nu,\ell,\vartheta} - C(\eta)\left|\bar{\chi}_{C(\eta)}g\right|^2_{\ell}.$$

Furthermore, for any $|\beta| \geqq 0$ *we have*

$$|\langle w^2(\ell,\vartheta)\partial_\beta[Kg_1],g_2\rangle| \leqq \left\{\eta\sum_{|\beta_1|\leqq|\beta|}\left|\partial_{\beta_1}g_1\right|_{\nu,\ell,\vartheta}+C(\eta)\left|\bar{\chi}_{C(\eta)}g_1\right|_{\ell}\right\}|g_2|_{\nu,\ell,\vartheta},$$

where we are using (21).

Some parts of the proof of Lemma 2 are exactly the same as in [10]. For instance, the proof of the lower bound for $\langle w^2(\ell,\vartheta)\partial_\beta[\nu g],\partial_\beta g\rangle$ is exactly the same. But the estimate for $|\langle w^2(\ell,\vartheta)\partial_\beta[Kg_1],g_2\rangle|$ requires extra care in particular because g_1 in the argument of $[Kg_1]$ does not depend only on v. We therefore need to control the new exponentially growing factor of $w(\ell,\vartheta)(v)$. This requires a close look at the variables from (22) in (23). In particular, we write down the analog of the conservation of energy (4) in this new coordinate system (31) in order absorb the exponentially growing weight. For completeness, we present all details of the proof.

Proof (The First Estimate in the Lemma).

Fix $\eta>0$. Recall

$$\langle w^2\partial_\beta[\nu g],\partial_\beta g\rangle=\langle w^2\nu\partial_\beta g,\partial_\beta g\rangle+\sum_{0<\beta_1\leqq\beta}C_\beta^{\beta_1}\langle w^2\partial_{\beta_1}\nu\partial_{\beta-\beta_1}g,\partial_\beta g\rangle.$$

By Lemma 2 in [9], for $|\beta_1|>0$,

$$|\partial_{\beta_1}\nu|\leqq C(1+|v|^2)^{\frac{\gamma-1}{2}}.$$

We use this estimate (for m is chosen large enough) to obtain (since $\gamma<0$)

$$\begin{aligned}\langle w^2\partial_{\beta_1}\nu\partial_{\beta-\beta_1}g,\partial_\beta g\rangle&=\int_{|v|\leqq m}+\int_{|v|\geqq m}\\&\leqq\int_{|v|\leqq m}+\frac{C}{m}\left|\partial_{\beta-\beta_1}g\right|_{\boldsymbol{\nu},\ell,\vartheta}\left|\partial_\beta g\right|_{\boldsymbol{\nu},\ell,\vartheta}\\&\leqq\int_{|v|\leqq m}+\frac{\eta}{2}\left|\partial_{\beta-\beta_1}g\right|_{\boldsymbol{\nu},\ell,\vartheta}\left|\partial_\beta g\right|_{\boldsymbol{\nu},\ell,\vartheta}.\end{aligned}$$

On the other hand, for such $m>0$ and $\beta-\beta_1<\beta$, the first integral over $|v|\leqq m$ is bounded by a compact Sobolev interpolation

$$\int_{|v|\leqq m}\leqq\frac{\eta}{2}\sum_{|\beta_1|=|\beta|}\left|\partial_{\beta_1}g\right|^2_{\boldsymbol{\nu},\ell}+C(\eta)\left|\bar{\chi}_{C(\eta)}g\right|^2_{\boldsymbol{\nu},\ell}.\tag{26}$$

This concludes the lower bound for $\langle w^2(\ell,\vartheta)\partial_\beta[\nu g],\partial_\beta g\rangle$.

The Second Estimate in the Lemma. The proof of the second estimate is divided into several parts. Recall $K=K_1-K_2$.

Step 1: The Estimate for K_1.

Next consider K_1 from (19). We change variables $u\to u+v$ to obtain

$$[K_1g_1](v)=\int_{\mathbb{R}^3\times S^2}B(\theta)|u|^\gamma\mu^{1/2}(u+v)\mu^{1/2}(v)g_1(u+v)du d\omega.$$

Notice that now $\cos\theta=u\cdot\omega/|u|$ so that for $|\beta|>0$, $\partial_\beta[K_1g_1](v)$

$$=\sum_{\beta_1\leqq\beta}C_\beta^{\beta_1}\int_{\mathbb{R}^3\times S^2}B(\theta)|u|^\gamma\partial_{\beta-\beta_1}\left(\mu^{1/2}(u+v)\mu^{1/2}(v)\right)\partial_{\beta_1}g_1(u+v)du d\omega.$$

For any $0 < q' < 1$ we have

$$|\partial_{\beta-\beta_1}\{\mu^{1/2}(u+v)\mu^{1/2}(v)\}| \leqq C(|\beta|, q')\mu^{q'/2}(u+v)\mu^{q'/2}(v).$$

We will use this exponential decay to control half of the exponential growth in the weight w. If $0 \leqq \vartheta < 2$ then

$$w(\ell,\vartheta)(v)\mu^{q'/2}(v) \leqq C\mu^{q'/4}(v).$$

If $\vartheta = 2$ then, for given $0 < q < 1$ in w, we choose q' so that $q < q' < 1$. And in this case

$$w(\ell,\vartheta)(v)\mu^{q'/2}(v) = (1+|v|^2)^{\tau\ell/2}e^{\frac{q}{4}}e^{\frac{q}{4}|v|^2}\mu^{q'/2}(v) \leqq C\mu^{(q'-q)/4}(v).$$

Choosing $0 < q'' < \min\{|q'-q|/4, q'/4\}$, we can always write $\langle w^2(\ell,\vartheta)\partial_\beta[K_1 g_1], g_2\rangle$

$$=\sum_{\beta_1\leqq\beta}\int_{\mathbb{R}^3\times\mathbb{R}^3} w(\ell,\vartheta)(v)|u|^\gamma \mu^{q''}(u+v)\mu^{q''}(v)\mu_{\beta_1}(u+v,v)\partial_{\beta_1}g_1(u+v)g_2(v)dudv,$$

where $\mu_{\beta_1}(u+v,v)$ is a collection of smooth functions satisfying

$$\left|\partial^u_{\bar\beta}\mu_{\beta_1}(u+v,v)\right| \leqq C(|\bar\beta|, q, q', q'').$$

Change variables $u \to u - v$ back to obtain

$$\sum_{\beta_1\leqq\beta}\int_{\mathbb{R}^3\times\mathbb{R}^3} w(\ell,\vartheta)(v)|u-v|^\gamma \mu^{q''}(u)\mu^{q''}(v)\mu_{\beta_1}(u,v)\partial_{\beta_1}g_1(u)g_2(v)dudv.$$

Now further split

$$\begin{aligned}\langle w^2(\ell,\vartheta)\partial_\beta[K_1 g_1], g_2\rangle &= \langle w^2(\ell,\vartheta)\partial_\beta[K_1^\chi g_1], g_2\rangle + \langle w^2(\ell,\vartheta)\partial_\beta[K_1^{1-\chi} g_1], g_2\rangle \\ &= \mathbf{K}_1^\chi + \mathbf{K}_1^{1-\chi}.\end{aligned}$$

Using (21) we have

$$\begin{aligned}\mathbf{K}_1^{1-\chi} \equiv \int & w(\ell,\vartheta)(v)\bar\chi_m(|u-v|)|u-v|^\gamma \mu^{q''}(u)\mu^{q''} \\ &\times(v)\mu_{\beta_1}(u,v)\partial_{\beta_1}g_1(u)g_2(v)dudv,\end{aligned}$$

where $\bar\chi_m = 1 - \chi_m$ and we implicitly sum over $\beta_1 \leqq \beta$. Then

$$\begin{aligned}\left|\mathbf{K}_1^{1-\chi}\right| \leqq C &\left\{\int w^2(\ell,\vartheta)(v)\bar\chi_m(|u-v|)|u-v|^\gamma \mu^{q''}(u)\mu^{q''}(v)\,|g_2(v)|^2\,dudv\right\}^{1/2} \\ &\times \sum_{\beta_1\leqq\beta}\left\{\int \bar\chi_m(|u-v|)|u-v|^\gamma\mu^{q''}(u)\mu^{q''}(v)\left|\partial_{\beta_1}g_1(u)\right|^2 dudv\right\}^{1/2}. \\ &\leqq C(2m)^{\frac{3+\gamma}{2}}|g_2|_{\nu,\ell,\vartheta}\sum_{\beta_1\leqq\beta}(2m)^{\frac{3+\gamma}{2}}|\partial_{\beta_1}g_1|_{\nu,\ell} \\ &\leqq \frac{\eta}{2}|g_2|_{\nu,\ell,\vartheta}\sum_{\beta_1\leqq\beta}|\partial_{\beta_1}g_1|_{\nu,\ell}.\end{aligned}$$

The last step follows by choosing m small enough.

Further,

$$\mathbf{K}_1^\chi \equiv \int w(\ell,\vartheta)(v)\chi_m(|u-v|)|u-v|^\gamma \mu^{q''}(u)\mu^{q''} \times(v)\mu_{\beta_1}(u,v)\partial_{\beta_1}g_1(u)g_2(v)dudv,$$

where we again implicitly sum over $\beta_1 \leqq \beta$. After an integration by parts

$$\mathbf{K}_1^\chi = \sum_{\beta_1\leqq\beta}(-1)^{|\beta_1|}\int w(\ell,\vartheta)(v)\partial^u_{\beta_1}\left\{\chi_m(|u-v|)|u-v|^\gamma\mu^{q''}(u)\mu_{\beta_1}(u,v)\right\} \times\mu^{q''}(v)g_1(u)g_2(v)dudv.$$

Since $|u-v|^\gamma$ is bounded now, from (21), choosing another $m'>0$ large enough we have

$$\begin{aligned}\left|\mathbf{K}_1^\chi\right| &\leqq C(|\beta|,m)\int w(\ell,\vartheta)(v)\mu^{q''/2}(u)\mu^{q''/2}(v)\,|g_1(u)g_2(v)|\,dudv\\ &=\int_{|u|\leqq m'}+\int_{|u|>m'}\\ &\leqq C\int w(\ell,\vartheta)(v)\bar{\chi}_{m'}(|u|)\mu^{q''/2}(u)\mu^{q''/2}(v)\,|g_1(u)g_2(v)|\,dudv\\ &\quad+Ce^{-\frac{q''}{8}m'}\int w(\ell,\vartheta)(v)\mu^{q''/4}(u)\mu^{q''/2}(v)\,|g_1(u)g_2(v)|\,dudv\\ &\leqq\left\{\frac{\eta}{2}|g_1|_{\nu,\ell}+C(m')|\bar{\chi}_{m'}g_1|_{\nu,\ell}\right\}|g_2|_{\nu,\ell,\vartheta}.\end{aligned}$$

This completes the estimate for $\langle w^2(\ell,\vartheta)\partial_\beta[K_1g_1],g_2\rangle$ and step one.

Step 2: The Estimate for K_2.

We turn to K_2 from (20). Split $K_2=K_2^\chi+K_2^{1-\chi}$ and consider K_2^χ in (22).

Step (2a): The Estimate of $K_2^{1-\chi}$.

Now consider $K_2^{1-\chi}=K_2-K_2^\chi$ which is given by

$$\begin{aligned}K_2^{1-\chi}g_1 \equiv &\int_{\mathbb{R}^3\times S^2}B(\theta)|u-v|^\gamma\mu^{1/2}(u)\bar{\chi}_m(|u-v|)\mu^{1/2}(u')g_1(v')dud\omega\\ &+\int_{\mathbb{R}^3\times S^2}B(\theta)|u-v|^\gamma\mu^{1/2}(u)\bar{\chi}_m(|u-v|)\mu^{1/2}(v')g_1(u')dud\omega.\end{aligned}$$

Here $\bar{\chi}_m=1-\chi_m$ and χ_m is defined in (21). Equation (3) and $\{|u-v|\leqq 2m\}$ imply

$$\begin{aligned}|u'|&=|v+u-v-[(u-v)\cdot\omega]\omega|\geqq|v|-2|u-v|\geqq|v|-4m,\\ |v'|&=|v+[(u-v)\cdot\omega]\omega|\geqq|v|-|u-v|\geqq|v|-2m.\end{aligned}$$

Therefore for any $0<q'<1$ we have

$$\mu^{1/2}(u)\mu^{1/2}(u')+\mu^{1/2}(u)\mu^{1/2}(v')\leqq e^{C(q')m^2}\mu^{1/2}(u)\mu^{q'/2}(v). \tag{27}$$

This will be the key point in estimating the $K_2^{1-\chi}$ part.

First we take a look at $\partial_\beta[K_2^{1-\chi} g_1]$. Change variables $u - v \to u$ to obtain

$$K_2^{1-\chi} g_1 \equiv \int_{\mathbb{R}^3\times S^2} B(\theta)|u|^\gamma \mu^{1/2}(u+v)\bar{\chi}_m(|u|)\mu^{1/2}(v+u_\perp)g_1(v+u_\parallel)dud\omega$$
$$+ \int_{\mathbb{R}^3\times S^2} B(\theta)|u|^\gamma \mu^{1/2}(u+v)\bar{\chi}_m(|u|)\mu^{1/2}(v+u_\parallel)g_1(v+u_\perp)dud\omega.$$

Note that $u_\parallel$ and $u_\perp$ are defined using notation from [10]:

$$u_\parallel \equiv [u\cdot\omega]\omega, \quad u_\perp \equiv u - [u\cdot\omega]\omega. \tag{28}$$

Now derivatives will not hit the singular kernel. $\partial_\beta[K_2^{1-\chi} g_1]$ is

$$C_\beta^{\beta_1}\int_{\mathbb{R}^3\times S^2} B(\theta)|u|^\gamma \bar{\chi}_m(|u|)\partial_{\beta-\beta_1}$$
$$\times\{\mu^{1/2}(u+v)\mu^{1/2}(v+u_\perp)\}\partial_{\beta_1} g_1(v+u_\parallel)dud\omega$$
$$+C_\beta^{\beta_1}\int_{\mathbb{R}^3\times S^2} B(\theta)|u|^\gamma \bar{\chi}_m(|u|)\partial_{\beta-\beta_1}$$
$$\times\{\mu^{1/2}(u+v)\mu^{1/2}(v+u_\parallel)\}\partial_{\beta_1} g_1(v+u_\perp)dud\omega,$$

where we implicitly sum over multi-indices $\beta_1 \leqq \beta$. Therefore, for any $0 < q'' < 1$, $|\partial_\beta[K_2^{1-\chi} g_1]|$ is bounded by

$$C\int_{\mathbb{R}^3\times S^2} |u|^\gamma \bar{\chi}_m(|u|)\mu^{q''/2}(u+v)\mu^{q''/2}(v+u_\perp)|\partial_{\beta_1} g_1(v+u_\parallel)|dud\omega$$
$$+C\int_{\mathbb{R}^3\times S^2} |u|^\gamma \bar{\chi}_m(|u|)\mu^{q''/2}(u+v)\mu^{q''/2}(v+u_\parallel)|\partial_{\beta_1} g_1(v+u_\perp)|dud\omega.$$

We change variables $u \to u - v$ back again to see that $|\partial_\beta[K_2^{1-\chi} g_1]|$ is bounded by

$$C\int_{\mathbb{R}^3\times S^2} |u-v|^\gamma \bar{\chi}_m(|u-v|)\mu^{q''/2}(u)\mu^{q''/2}(v')|\partial_{\beta_1} g_1(u')|dud\omega$$
$$+C\int_{\mathbb{R}^3\times S^2} |u-v|^\gamma \bar{\chi}_m(|u-v|)\mu^{q''/2}(u)\mu^{q''/2}(u')|\partial_{\beta_1} g_1(v')|dud\omega.$$

Use (27) for any $0 < q' < q''$ to say this is bounded above by

$$C\int_{\mathbb{R}^3\times S^2} |u-v|^\gamma \bar{\chi}_m(|u-v|)\mu^{q''/2}(u)\mu^{q'/2}(v)\left\{|\partial_{\beta_1} g_1(u')| + |\partial_{\beta_1} g_1(v')|\right\}dud\omega.$$

We remark that this last bound is true (and trivial) when $|\beta| = 0$ in which case $\partial_{\beta_1} = \partial_0 = 1$ by convention. Thus, $|\langle w^2(\ell,\vartheta)\partial_\beta\{K_2^{1-\chi} g_1\}, g_2\rangle|$ is

$$\leqq C\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} |u-v|^\gamma \bar{\chi}_m(|u-v|)w^2(\ell,\vartheta)(v)\mu^{q''/2}(u)\mu^{q'/2}(v)$$
$$\times |\partial_{\beta_1} g_1(v')||g_2(v)|,$$
$$+ C\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} |u-v|^\gamma \bar{\chi}_m(|u-v|)w^2(\ell,\vartheta)(v)\mu^{q''/2}(u)\mu^{q'/2}(v)$$
$$\times |\partial_{\beta_1} g_1(u')||g_2(v)|.$$

Here again we need to control the large exponentially growing factor $w(\ell, \vartheta)(v)$ by the strong exponential decay of the Maxwellians.

We will control this growth in these cases. If $0 \leqq \vartheta < 2$ then $q' > 0$ means

$$w(\ell, \vartheta)(v)\mu^{q'/2}(v) \leqq C\mu^{q'/4}(v).$$

Alternatively, if $\vartheta = 2$ with $0 < q < 1$ then we can choose q' and q'' such that $q < q' < q''$. Then

$$w(\ell, \vartheta)(v)\mu^{q'/2}(v) \leqq Cw(\ell, 0)(v)\mu^{(q'-q)/2}(v) \leqq C\mu^{(q'-q)/4}(v).$$

In either case, choose $q_1 = \min\{q''/2, q'/4, |q' - q|/4\} > 0$. Then we have the upper bound of

$$C\int |u - v|^{\gamma}\bar{\chi}_m(|u - v|)w(\ell, \vartheta)(v)\mu^{q_1}(u)\mu^{q_1}(v)|\partial_{\beta_1}g_1(v')||g_2(v)|dvdud\omega$$
$$+C\int |u - v|^{\gamma}\bar{\chi}_m(|u - v|)w(\ell, \vartheta)(v)\mu^{q_1}(u)\mu^{q_1}(v)|\partial_{\beta_1}g_1(u')||g_2(v)|dvdud\omega.$$

Further note that

$$\int |u - v|^{\gamma}\bar{\chi}_m(|u - v|)\mu^{q_1}(u)du \leqq Cm^{3+\gamma}.$$

Apply Cauchy–Schwartz and the last display to obtain the upper bound

$$\leqq Cm^{\frac{3+\gamma}{2}}\left\{\iint |u - v|^{\gamma}\bar{\chi}_m(|u-v|)\mu^{q_1}(u)\mu^{q_1}(v)|\partial_{\beta_1}g_1(v')|^2dvdud\omega\right\}^{1/2}|g_2|_{\mathbf{v},\ell,\vartheta}$$
$$+Cm^{\frac{3+\gamma}{2}}\left\{\iint |u-v|^{\gamma}\bar{\chi}_m(|u - v|)\mu^{q_1}(u)\mu^{q_1}(v)|\partial_{\beta_1}g_1(u')|^2dvdud\omega\right\}^{1/2}|g_2|_{\mathbf{v},\ell,\vartheta}.$$

Now apply the change of variables $(u, v) \to (u', v')$ using $|u - v| = |u' - v'|$ and (4) to see that $|\langle w^2(\ell, \vartheta)\partial_\beta\{K_2^{1-\chi}g_1\}, g_2\rangle|$ is bounded by

$$Cm^{3+\gamma}\left\{\iint |u - v|^{\gamma}\bar{\chi}_m(|u - v|)\mu^{q_1}(u)\mu^{q_1}(v)|\partial_{\beta_1}g_1(v)|^2dvdu\right\}^{1/2}|g_2|_{\mathbf{v},\ell,\vartheta}.$$

Hence,

$$|\langle w^2(\ell, \vartheta)\partial_\beta\{K_2^{1-\chi}g_1\}, g_2\rangle| \leqq Cm^{\gamma+3}|g_2|_{\mathbf{v},\ell,\vartheta}\sum_{|\beta_1|\leqq|\beta|}|\partial_{\beta_1}g_1|_{\mathbf{v},\ell}.$$

For $m > 0$ small enough, we have completed the estimate of $K_2^{1-\chi}$, step (2a).

Step (2b): Estimate of K_2^{χ}.

For some large but fixed $m' > 0$ we define the smooth cutoff function

$$\Upsilon_{m'} = \Upsilon_{m'}(v, \xi) = \chi_{m'}(\sqrt{1 + |v|^2 + |v + \xi|^2}), \quad \bar{\Upsilon}_{m'} = 1 - \Upsilon_{m'}(v, \xi),$$

where $\chi_{m'}$ is defined in (21). Now split again $K_2^{\chi} = K_2^{\Upsilon} + K_2^{1-\Upsilon}$, where

$$K_2^{\Upsilon}g_1 = \int_{\mathbb{R}^3}\Upsilon_{m'}(v, \xi)k_2^{\chi}(v, \xi)g_1(v + \xi)d\xi.$$

We will estimate this term first. Taking derivatives

$$\partial_\beta[K_2^\Upsilon g_1] = \sum_{\beta_1 \leqq \beta} C_\beta^{\beta_1} \int_{\mathbb{R}^3} \partial_{\beta_1}^v [\Upsilon_{m'}(v,\xi) k_2^\chi(v,\xi)] \partial_{\beta-\beta_1} g_1(v+\xi) d\xi.$$

Using Lemma 1, with $0 < s_1 < s_2 < 1$, $|\langle w^2(\ell,\vartheta)\partial_\beta[K_2^\Upsilon g_1], g_2\rangle|$ is bounded by

$$C \sum_{\beta_1 \leqq \beta} \int_{|v|+|v+\xi|>m'} \frac{w^2(\ell,\vartheta)(v)|\partial_{\beta-\beta_1} g_1(v+\xi)||g_2(v)|}{|\xi|(1+|v|+|v+\xi|)^{1-\gamma}} e^{-\frac{s_2}{8}|\xi|^2 - \frac{s_1}{2}|\zeta_{||}|^2} d\xi dv. \tag{29}$$

By (10) we expand

$$w(\ell,\vartheta)(v) e^{-\frac{s_2}{8}|\xi|^2 - \frac{s_1}{2}|\zeta_{||}|^2} = (1+|v|^2)^{\tau\ell/2} e^{\frac{q}{4}(1+|v|^2)^{\vartheta/2}} e^{-\frac{s_2}{8}|\xi|^2 - \frac{s_1}{2}|\zeta_{||}|^2}. \tag{30}$$

If we can control (30) by $w(\ell,\vartheta)(v+\xi)$ times decay in other directions then we can estimate (29). To do this, we look for an analog of (4) in the variables (23).

Using (23) we have

$$|v|^2 + |v+\xi-\zeta_\perp|^2 = |v+\xi|^2 + |v-\zeta_\perp|^2 = |v+\xi|^2 + \left(v \cdot \frac{\xi}{|\xi|}\right)^2. \tag{31}$$

Since $0 \leqq \vartheta \leqq 2$ we have

$$|v|^\vartheta \leqq \left(|v+\xi|^2 + \left(v\cdot\frac{\xi}{|\xi|}\right)^2\right)^{\vartheta/2} \leqq |v+\xi|^\vartheta + \left|v\cdot\frac{\xi}{|\xi|}\right|^\vartheta.$$

Thus,

$$e^{\frac{q}{4}(1+|v|^2)^{\vartheta/2}} \leqq e^{\frac{q}{4}} e^{\frac{q}{4}|v|^\vartheta} \leqq e^{\frac{q}{4}} e^{\frac{q}{4}|v+\xi|^\vartheta} e^{\frac{q}{4}\left|v\cdot\frac{\xi}{|\xi|}\right|^\vartheta}. \tag{32}$$

Further, from (23) notice that

$$|\zeta_{\|}|^2 = \left(\left(v\cdot\frac{\xi}{|\xi|}\right) + \frac{1}{2}|\xi|\right)^2 \geqq \frac{1}{2}\left(v\cdot\frac{\xi}{|\xi|}\right)^2 - \frac{1}{4}|\xi|^2.$$

Therefore with $0 < s_1 < s_2$ we have

$$e^{-\frac{s_2}{8}|\xi|^2 - \frac{s_1}{2}|\zeta_{||}|^2} \leqq e^{-\frac{s_2-s_1}{8}|\xi|^2 - \frac{s_1}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}.$$

Combine the above with (32), to obtain

$$\begin{aligned} e^{\frac{q}{4}(1+|v|^2)^{\vartheta/2}} e^{-\frac{s_2}{8}|\xi|^2 - \frac{s_1}{2}|\zeta_{||}|^2} &\leqq e^{\frac{q}{4}} e^{\frac{q}{4}|v+\xi|^\vartheta} e^{\frac{q}{4}\left|v\cdot\frac{\xi}{|\xi|}\right|^\vartheta} e^{-\frac{s_2}{8}|\xi|^2 - \frac{s_1}{2}|\zeta_{||}|^2} \\ &\leqq C e^{\frac{q}{4}|v+\xi|^\vartheta} e^{\frac{q}{4}\left|v\cdot\frac{\xi}{|\xi|}\right|^\vartheta} e^{-\frac{s_2-s_1}{8}|\xi|^2 - \frac{s_1}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2} \\ &= C e^{\frac{q}{4}|v+\xi|^\vartheta} e^{-\frac{s_2-s_1}{8}|\xi|^2} e^{\frac{q}{4}\left|v\cdot\frac{\xi}{|\xi|}\right|^\vartheta - \frac{s_1}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}. \end{aligned}$$

If $0 \leqq \vartheta < 2$ then

$$e^{\frac{q}{4}\left|v\cdot\frac{\xi}{|\xi|}\right|^{\vartheta}-\frac{s_1}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2} \leqq Ce^{-\frac{s_1}{8}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}.$$

And if $\vartheta = 2$, then $0 < q < 1$ and we can choose s_1 with $1 > s_1 > q$ so that

$$e^{\frac{q}{4}\left|v\cdot\frac{\xi}{|\xi|}\right|^{2}-\frac{s_1}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2} \leqq Ce^{-\frac{s_1-q}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}.$$

In either case, choosing $s_3 = \min\{|s_1 - q|, s_1/2\} > 0$ and plugging these estimates into (30), we conclude that

$$\begin{aligned} &w(\ell,\vartheta)(v)e^{-\frac{s_2}{8}|\xi|^2-\frac{s_1}{2}|\zeta_{||}|^2} \\ &\leqq C(1+|v|^2)^{\tau\ell/2}e^{\frac{q}{4}(1+|v+\xi|^2)^{\vartheta/2}}e^{-\frac{s_2-s_1}{8}|\xi|^2-\frac{s_3}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}. \end{aligned} \tag{33}$$

Next, we estimate $(1+|v|^2)^{\tau\ell/2}$ with $\tau < 0$ and $\ell \in \mathbb{R}$. If $\ell\tau > 0$ then (31) yields

$$\begin{aligned} (1+|v|^2)^{\tau\ell/2} &\leqq \left(1+|v+\xi|^2+\left(v\cdot\frac{\xi}{|\xi|}\right)^2\right)^{\tau\ell/2} \\ &\leqq C(1+|v+\xi|^2)^{\tau\ell/2}\left(1+\left(v\cdot\frac{\xi}{|\xi|}\right)^2\right)^{\tau\ell/2}. \end{aligned}$$

Conversely if $\ell\tau \leqq 0$ then we split the region into

$$\{|v+\xi| > 2|v|\} \cup \{|v+\xi| \leqq 2|v|\}.$$

On $\{|v+\xi| \leqq 2|v|\}$ and $\ell\tau \leqq 0$ then

$$(1+|v|^2)^{\tau\ell/2} \leqq C(1+|v+\xi|^2)^{\tau\ell/2}.$$

Alternatively, if $\{|v+\xi| > 2|v|\}$ then

$$|\xi| \geqq |v+\xi| - |v| > |v+\xi|/2.$$

We therefore have

$$(1+|v|^2)^{\tau\ell/2}e^{-\frac{s_2-s_1}{8}|\xi|^2} \leqq e^{-\frac{s_2-s_1}{8}|\xi|^2} \leqq e^{-\frac{s_2-s_1}{16}|\xi|^2}e^{-\frac{s_2-s_1}{64}|v+\xi|^2}.$$

In any of these last few cases, since $s_2 > s_1 > q$, from (33) we can conclude

$$\begin{aligned} w(\ell,\vartheta)(v)e^{-\frac{s_2}{8}|\xi|^2-\frac{s_1}{2}|\zeta_{||}|^2} &\leqq C(1+|v|^2)^{\tau\ell/2}e^{\frac{q}{4}(1+|v+\xi|^2)^{\vartheta/2}}e^{-\frac{s_2-s_1}{8}|\xi|^2-\frac{s_3}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2} \\ &\leqq C(1+|v+\xi|^2)^{\tau\ell/2}e^{\frac{q}{4}(1+|v+\xi|^2)^{\vartheta/2}}e^{-\frac{s_2-s_1}{16}|\xi|^2-\frac{s_3}{8}\left(v\cdot\frac{\xi}{|\xi|}\right)^2} \\ &= Cw(\ell,\vartheta)(v+\xi)e^{-\frac{s_2-s_1}{16}|\xi|^2-\frac{s_3}{8}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}. \end{aligned}$$

Plug this into (29) to obtain the following upper bound for (29) of

$$C\int_{|v|+|v+\xi|>m'} \frac{\left(w(\ell,\vartheta)(v+\xi)|\partial_{\beta-\beta_1}g_1(v+\xi)|\right)\left(w(\ell,\vartheta)(v)|g_2(v)|\right)}{|\xi|(1+|v|+|v+\xi|)^{1-\gamma}} \times e^{-\frac{s_2-s_1}{16}|\xi|^2}d\xi dv,$$

where we implicitly sum over $\beta_1 \leqq \beta$. Using Cauchy–Schwartz and translation invariance this is

$$\begin{aligned}
&\leqq \frac{C}{m'}\int \frac{\left(w(\ell,\vartheta)(v+\xi)|\partial_{\beta-\beta_1}g_1(v+\xi)|\right)\left(w(\ell,\vartheta)(v)|g_2(v)|\right)}{|\xi|(1+|v|+|v+\xi|)^{-\gamma}}e^{-\frac{s_2-s_1}{16}|\xi|^2}d\xi dv \\
&\quad\leqq \frac{C}{m'}|g_2|_{\nu,\ell,\vartheta}\int \frac{w^2(\ell,\vartheta)(v+\xi)|\partial_{\beta-\beta_1}g_1(v+\xi)|^2}{|\xi|(1+|v+\xi|)^{-\gamma}}e^{-\frac{s_2-s_1}{16}|\xi|^2}d\xi dv \\
&\quad\leqq \frac{C}{m'}\sum_{\beta_1\leqq\beta}|\partial_{\beta-\beta_1}g_1|_{\nu,\ell,\vartheta}|g_2|_{\nu,\ell,\vartheta}.
\end{aligned}$$

This completes the estimate for K_2^{Υ}.

We now estimate $K_2^{1-\Upsilon}$. Taking derivatives

$$\partial_\beta[K_2^{1-\Upsilon}g_1] = \sum_{\beta_1\leqq\beta} C_\beta^{\beta_1}\int_{\mathbb{R}^3}\partial_{\beta_1}^v[\bar{\Upsilon}_{m'}(v,\xi)k_2^\chi(v,\xi)]\partial_{\beta-\beta_1}g_1(v+\xi)d\xi.$$

Also,

$$\langle w^2\partial_\beta[K_2^{1-\Upsilon}g_1], g_2\rangle = \int w^2(\ell,\vartheta)\partial_\beta[K_2^{1-\Upsilon}g_1]g_2(v)dv.$$

By Cauchy–Schwartz and the compact support of $K_2^{1-\Upsilon}$ we have

$$\left|\langle w^2\partial_\beta[K_2^{1-\Upsilon}g_1], g_2\rangle\right| \leqq C(m')\left\{\int_{|v|\leqq m'}\left(\partial_\beta[K_2^{1-\Upsilon}g_1]\right)^2 dv\right\}^{1/2}|g_2|_{\nu,\ell}.$$

With Lemma 1 we established that $\partial_\beta[K_2^{1-\Upsilon}g_1]$ is compact from H^k to H^k. Then by the general interpolation for compact operators from H^k to H^k we have

$$\left|\langle w^2\partial_\beta[K_2^{1-\Upsilon}g_1], g_2\rangle\right| \leqq \left\{\frac{\eta}{4}\sum_{|\beta_1|=|\beta|}\left|\partial_{\beta_1}g_1\right|_{\nu,\ell} + C(\eta,m')\left|g_1\right|_{\nu,\ell}\right\}|g_2|_{\nu,\ell}.$$

This completes the estimate for $K_2^{1-\Upsilon}$ and thus for K_2^χ, step (2b). We have therefore finished the whole proof. □

The following Corollary is used to prove existence of global solutions.

Corollary 1. *Let* $|\beta| > 0$, $\ell \in \mathbb{R}$, $0 \leqq \vartheta \leqq 2$ *and* $0 < q$. *If* $\vartheta = 2$ *restrict* $0 < q < 1$. *Then for all* $\eta > 0$ *there exists* $C(\eta) > 0$ *such that*

$$\langle w^2\partial_\beta[Lg], \partial_\beta g\rangle \geqq \left|\partial_\beta g\right|^2_{\nu,\ell,\vartheta} - \eta\sum_{|\beta_1|\leqq|\beta|}\left|\partial_{\beta_1}g\right|^2_{\nu,\ell,\vartheta} - C(\eta)\left|\bar{\chi}_{C(\eta)}g\right|^2_\ell,$$

where $\bar{\chi}_{C(\eta)}$ *is from* (21).

The rest of this section is devoted to estimates for the nonlinear collision term $\Gamma[g_1, g_2]$, with $g_i(x, v)$ $(i = 1, 2)$. In (5), the (nonsymmetric) bilinear form $\Gamma[g_1, g_2]$ in the Boltzmann case is

$$\begin{aligned}\Gamma[g_1, g_2] &= \mu^{-1/2}(v) Q[\mu^{1/2} g_1, \mu^{1/2} g_2] \equiv \Gamma_{\text{gain}}[g_1, g_2] - \Gamma_{\text{loss}}[g_1, g_2],\\ &= \int_{\mathbb{R}^3} |u - v|^\gamma \mu^{1/2}(u) \left[\int_{S^2} B(\theta) g_1(u') g_2(v') d\omega\right] du, \qquad (34)\\ &\quad - g_2(v) \int_{\mathbb{R}^3} |u - v|^\gamma \mu^{1/2}(u) \left[\int_{S^2} B(\theta) d\omega\right] g_1(u) du.\end{aligned}$$

The change of variables $u - v \to u$ gives

$$\begin{aligned}\partial^\alpha_\beta \Gamma[g_1, g_2] &\equiv \partial^\alpha_\beta \left[\int_{\mathbb{R}^3} \int_{S^2} |u|^\gamma \mu^{1/2}(u + v) g_1(v + u_\parallel) g_2(v + u_\perp) B(\theta) du d\omega\right],\\ &\quad - \partial^\alpha_\beta \left[\int_{\mathbb{R}^3} \int_{S^2} |u|^\gamma \mu^{1/2}(u + v) g_1(v + u) g_2(v) B(\theta) du d\omega\right],\\ &\equiv \sum C^{\beta_0 \beta_1 \beta_2}_\beta C^{\alpha_1 \alpha_2}_\alpha \Gamma^0[\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2],\end{aligned}$$

where the summation is over $\beta_0 + \beta_1 + \beta_2 = \beta$ and $\alpha_1 + \alpha_2 = \alpha$. Also $u_\perp$, $u_\parallel$ are given by (28). By the product rule and the reverse change of variables we have

$$\begin{aligned}\Gamma^0[\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2], &\equiv \int_{\mathbb{R}^3} \int_{S^2} |u - v|^\gamma \partial_{\beta_0}[\mu^{1/2}(u)] \partial^{\alpha_1}_{\beta_1} g_1(u') \partial^{\alpha_2}_{\beta_2} g_2(v') B(\theta)\\ &\quad - \partial^{\alpha_2}_{\beta_2} g_2(v) \int_{\mathbb{R}^3} \int_{S^2} |u - v|^\gamma \partial_{\beta_0}[\mu^{1/2}(u)] \partial^{\alpha_1}_{\beta_1} g_1(u) B(\theta)\\ &\equiv \Gamma^0_{\text{gain}} - \Gamma^0_{\text{loss}}. \qquad (35)\end{aligned}$$

With these formulas, we have the following nonlinear estimate:

Lemma 3. *Recall* (35) *and let* $\beta_0 + \beta_1 + \beta_2 = \beta$, $\alpha_1 + \alpha_2 = \alpha$. *Say* $0 \leqq \vartheta \leqq 2$, $0 < q$ *. If* $\vartheta = 2$, *restrict* $0 < q < 1$. *Let* $\ell = |\beta| - l$ *with* $l \geqq 0$. *If* $|\alpha_1| + |\beta_1| \leqq N/2$, *then*

$$|(w^2(\ell, \vartheta) \Gamma^0[\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2],\ \partial^\alpha_\beta g_3)| \leqq C \mathcal{E}^{1/2}_{l,\vartheta}(g_1) ||\partial^{\alpha_2}_{\beta_2} g_2||_{\nu, |\beta_2| - l, \vartheta} ||\partial^\alpha_\beta g_3||_{\nu, \ell, \vartheta}.$$

Alternatively, if $|\alpha_2| + |\beta_2| \leqq N/2$, *then*

$$|(w^2(\ell, \vartheta) \Gamma^0[\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2],\ \partial^\alpha_\beta g_3)| \leqq C ||\partial^{\alpha_1}_{\beta_1} g_1||_{\nu, |\beta_1| - l, \vartheta} \mathcal{E}^{1/2}_{l,\vartheta}(g_2) ||\partial^\alpha_\beta g_3||_{\nu, \ell, \vartheta}.$$

The proof of Lemma 3 is more or less the same as in [10]. However, small modifications are needed to facilitate the exponentially growing weight. In (39) we need to use (4) to properly distribute the exponentially growing factor in $w^2(\ell, \vartheta)(v)$.

Proof.

Case 1: *The Loss Term Estimate*

First consider the second term Γ^0_{loss} in (35). Note that

$$|\partial_{\beta_0}[\mu^{1/2}(u)]| \leqq C e^{-|u|^2/8}.$$

With $|\alpha_1|+|\beta_1|\leqq N/2$ and $\gamma>-3$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}|u-v|^\gamma|\partial_{\beta_0}[\mu^{1/2}(u)]\partial^{\alpha_1}_{\beta_1}g_1(x,u)|du\\
&\leqq C\left\{\int_{\mathbb{R}^3}|u-v|^\gamma e^{-|u|^2/8}|\partial^{\alpha_1}_{\beta_1}g_1(x,u)|^2du\right\}^{1/2}\left\{\int_{\mathbb{R}^3}|u-v|^\gamma e^{-|u|^2/8}du\right\}^{1/2}\\
&\leqq C\sup_{x,u}\left|e^{-|u|^2/16}\partial^{\alpha_1}_{\beta_1}g_1(x,u)\right|\left\{\int_{\mathbb{R}^3}|u-v|^\gamma e^{-|u|^2/16}du\right\}\\
&\leqq C\mathcal{E}_l^{1/2}(g_1)[1+|v|]^\gamma.
\end{aligned}$$

Since $N\geqq 8$, we have used the embedding $H^4(\mathbb{T}^3\times\mathbb{R}^3)\subset L^\infty$ to argue that

$$\sup_{x,u}\left|e^{-|u|^2/16}\partial^{\alpha_1}_{\beta_1}g_1(x,u)\right|\leqq C\mathcal{E}_l^{1/2}(g_1), \tag{36}$$

where $\mathcal{E}_l^{1/2}(g_1)$ is defined in (13). Hence $\left|(w^2(\ell,\vartheta)\Gamma^0_{\text{loss}}[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2],\partial^\alpha_\beta g_3)\right|$ is bounded by

$$\begin{aligned}
C\mathcal{E}_l^{1/2}(g_1)&\int[1+|v|]^\gamma w^2(\ell,\vartheta)(v)|\partial^{\alpha_2}_{\beta_2}g_2(v)\partial^\alpha_\beta g_3(v)|dvdx\\
&\leqq C\mathcal{E}_l^{1/2}(g_1)||\partial^{\alpha_2}_{\beta_2}g_2||_{\boldsymbol{\nu},\ell,\vartheta}||\partial^\alpha_\beta g_3||_{\boldsymbol{\nu},\ell,\vartheta}.
\end{aligned}$$

This completes the estimate for Γ^0_{loss} when $|\alpha_1|+|\beta_1|\leqq N/2$.

Now consider Γ^0_{loss} with $|\alpha_2|+|\beta_2|\leqq N/2$. Here the (u,v) integration domain is split into three parts

$$\{|v-u|\leqq|v|/2\}\cup\ \{|v-u|\geqq|v|/2,\quad |v|\geqq 1\}\cup\{|v-u|\geqq|v|/2,\quad |v|\leqq 1\}.$$

In the first region, $|u|$ is comparable to $|v|$ and thus we can use exponential decay in both variables to get the estimate. In the second and third regions, $|u|$ is not comparable to $|v|$ but we exploit the largeness or smallness of $|v|$ to get the estimate.

Case (1a): *The Loss Term in the First Region* $\{|v-u|\leqq|v|/2\}$.

For the first part, $\{|v-u|\leqq|v|/2\}$, we have

$$|u|\geqq|v|-|v-u|\geqq|v|/2.$$

So that

$$e^{-|u|^2/8}\leqq e^{-|u|^2/16}e^{-|v|^2/64}.$$

Then the integral of $w^2\Gamma^0_{\text{loss}}[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2]\partial^\alpha_\beta g_3$ over $\{|u-v|\leqq|v|/2\}$ is bounded by

$$\begin{aligned}
&C\int|u-v|^\gamma e^{-|u|^2/16}e^{-|v|^2/64}w^2(\ell,\vartheta)(v)|\partial^{\alpha_1}_{\beta_1}g_1(u)\partial^{\alpha_2}_{\beta_2}g_2(v)\partial^\alpha_\beta g_3(v)|dudvdx\\
&\leqq C\left\{\int|u-v|^\gamma e^{-|u|^2/16}e^{-|v|^2/64}|\partial^{\alpha_1}_{\beta_1}g_1(u)|^2dudvdx\right\}^{1/2}\\
&\times\left\{\int|u-v|^\gamma e^{-|u|^2/16}e^{-|v|^2/64}w^4(\ell,\vartheta)(v)|\partial^{\alpha_2}_{\beta_2}g_2(v)|^2|\partial^\alpha_\beta g_3(v)|^2dudvdx\right\}^{1/2}.
\end{aligned}$$

Integrating over dv, the first factor is bounded by $C||\partial_{\beta_1}^{\alpha_1} g_1||_{\boldsymbol{v},\ell}$. Integrating first over u variables in the second factor yields an upper bound

$$
\begin{aligned}
&C\left\{\int\left(\int|u-v|^{\gamma} e^{-|u|^2/16}du\right)e^{-|v|^2/64}w^4|\partial_{\beta_2}^{\alpha_2} g_2|^2|\partial_{\beta}^{\alpha} g_3|^2 dvdx\right\}^{1/2}\\
&\leqq C\left\{\int\int e^{-|v|^2/64}w^4(\ell,\vartheta)(v)[1+|v|]^{\gamma}|\partial_{\beta_2}^{\alpha_2} g_2|^2|\partial_{\beta}^{\alpha} g_3|^2 dvdx\right\}^{1/2}.
\end{aligned}
$$

And as in (36), since $N \geqq 8$ and $|\alpha_2|+|\beta_2| \leqq N/2$, by (13), we have

$$
\sup_{x,v} w(\ell,\vartheta)(v)e^{-|v|^2/256}|\partial_{\beta_2}^{\alpha_2} g_2(x,v)| \leqq C\mathcal{E}_{l,\vartheta}^{1/2}(g_2). \tag{37}
$$

We thus conclude the estimate over the first region.

Case (1b): *The Loss Term in the Second Region* $\{|v-u| \geqq |v|/2,\ \ |v| \leqq 1\}$.

Next consider $\Gamma_{\rm loss}^0$ over the second region $\{|v-u| \geqq |v|/2,\ \ |v| \geqq 1\}$. Since $\gamma < 0$, we have

$$
|u-v|^{\gamma} \leqq C[1+|v|]^{\gamma}.
$$

Then the integral of $w^2(\ell,\vartheta)\Gamma_{\rm loss}^0[\partial_{\beta_1}^{\alpha_1} g_1, \partial_{\beta_2}^{\alpha_2} g_2]\partial_{\beta}^{\alpha} g_3$ over this region is bounded by

$$
\begin{aligned}
&C\int[1+|v|]^{\gamma}e^{-|u|^2/8}w^2(\ell,\vartheta)|\partial_{\beta_1}^{\alpha_1} g_1(u)\partial_{\beta_2}^{\alpha_2} g_2(v)\partial_{\beta}^{\alpha} g_3(v)|dudvdx\\
&\leqq C\int\left\{\int e^{-|u|^2/8}|\partial_{\beta_1}^{\alpha_1} g_1(u)|du\right\}\\
&\quad\times\left\{\int[1+|v|]^{\gamma}w^2(\ell,\vartheta)|\partial_{\beta_2}^{\alpha_2} g_2(v)\partial_{\beta}^{\alpha} g_3(v)|dv\right\}dx\\
&\leqq C\int|\partial_{\beta_1}^{\alpha_1} g_1|_{\boldsymbol{v},\ell}\left\{\int w^2|\partial_{\beta_2}^{\alpha_2} g_2|^2dv\right\}^{1/2}\left\{\int[1+|v|]^{2\gamma}w^2|\partial_{\beta}^{\alpha} g_3|^2dv\right\}^{1/2}dx.
\end{aligned}
$$

Since $|\alpha_2|+|\beta_2| \leqq N/2$, $N \geqq 8$ and $H^2(\mathbb{T}^3) \subset L^{\infty}(\mathbb{T}^3)$, we have

$$
\sup_{x}\int w^2(\ell,\vartheta)(v)|\partial_{\beta_2}^{\alpha_2} g_2(x,v)|^2dv \leqq C\mathcal{E}_{l,\vartheta}(g_2). \tag{38}
$$

Thus, by the Cauchy–Schwartz inequality, the $\Gamma_{\rm loss}^0$ term over $\{|v-u| \geqq |v|/2,\ \ |v| \geqq 1\}$ is bounded by $C||\partial_{\beta_1}^{\alpha_1} g_1||_{\boldsymbol{v},\ell}\mathcal{E}_{l,\vartheta}^{1/2}(g_2)||\partial_{\beta}^{\alpha} g_3||_{\boldsymbol{v},\ell,\vartheta}$.

Case (1c): *The Loss Term in the Third Region* $\{|v-u| \geqq |v|/2,\ \ |v| \leqq 1\}$.

For the last region, $\{|v-u| \geqq |v|/2,\ \ |v| \leqq 1\}$, we have

$$
|u-v|^{\gamma} \leqq C|v|^{\gamma},\quad w(\ell,\vartheta)(v) \leqq C.
$$

Then the integral of $w^2(\ell,\vartheta)\Gamma^0_{\text{loss}}[\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2]\partial^{\alpha}_{\beta} g_3$ over this region is bounded by

$$
\begin{aligned}
C\int_{\{|v-u|\geqq|v|/2,\ \ |v|\leqq 1\}} &|u-v|^{\gamma} e^{-|u|^2/8}|\partial^{\alpha_1}_{\beta_1} g_1(u)\partial^{\alpha_2}_{\beta_2} g_2(v)\partial^{\alpha}_{\beta} g_3(v)|du dv dx \\
&\leqq C\int\left\{\int |u-v|^{\gamma/2} e^{-|u|^2/8}|\partial^{\alpha_1}_{\beta_1} g_1(u)|du\right\} \\
&\quad\times\left\{\int_{|v|\leqq 1}|v|^{\gamma/2}|\partial^{\alpha_2}_{\beta_2} g_2(v)\partial^{\alpha}_{\beta} g_3(v)|dv\right\} dx.
\end{aligned}
$$

Using the Cauchy–Schwartz inequality a few times we have

$$
\begin{aligned}
&\leqq C\int\left\{\int |u-v|^{\gamma} e^{-|u|^2/8}du\right\}^{1/2}\left\{\int e^{-|u|^2/8}|\partial^{\alpha_1}_{\beta_1} g_1(u)|^2 du\right\}^{1/2} \\
&\quad\times\left\{\int_{|v|\leqq 1}|v|^{\gamma}|\partial^{\alpha_2}_{\beta_2} g_2|^2 dv\right\}^{1/2}\left\{\int_{|v|\leqq 1}|\partial^{\alpha}_{\beta} g_3|^2 dv\right\}^{1/2} dx \\
&\leqq C\int |\partial^{\alpha_1}_{\beta_1} g_1|_{\nu,\ell}\left\{\int_{|v|\leqq 1}|v|^{\gamma}|\partial^{\alpha_2}_{\beta_2} g_2|^2 dv\right\}^{1/2}\left\{\int_{|v|\leqq 1}|\partial^{\alpha}_{\beta} g_3|^2 dv\right\}^{1/2} dx.
\end{aligned}
$$

By $|\alpha_2|+|\beta_2|\leqq N/2$, $\gamma>-3$ and $H^4(\mathbb{T}^3\times\mathbb{R}^3)\subset L^\infty$,

$$
\int_{|v|\leqq 1}|v|^{\gamma}|\partial^{\alpha_2}_{\beta_2} g_2|^2 dv \leq C\sup_{|v|\leqq 1, x\in\mathbb{T}^3}|\partial^{\alpha_2}_{\beta_2} g_2|^2 \leqq C\mathcal{E}_l(g_2).
$$

Hence, by Cauchy–Schwartz, the last part is bounded by

$$
C||\partial^{\alpha_1}_{\beta_1} g_1||_{\nu,\ell}\mathcal{E}_l^{1/2}(g_2)||\partial^{\alpha}_{\beta} g_3||_{\nu,\ell}.
$$

This concludes the desired estimate for Γ^0_{loss}.

Case 2: *The Gain Term Estimate.*

The next step is to estimate the gain term Γ^0_{gain} in (35), for which the (u,v) integration domain is split into two parts

$$
\{|u|\geqq|v|/2\}\ \cup\ \{|u|\leqq|v|/2\}.
$$

Case (2a) *The Gain Term over* $\{|u|\geqq|v|/2\}$.

For the first region $\{|u|\geqq|v|/2\}$,

$$
e^{-|u|^2/8}\leqq e^{-|u|^2/16}e^{-|v|^2/64}.
$$

Then the integral of $w^2(\ell,\vartheta)\Gamma^0_{\text{gain}}[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2]\partial^{\alpha_3}_{\beta_3}g_3$ over $\{|u|\geqq|v|/2\}$ is thus bounded by

$$\begin{aligned}&\int_{|u|\geqq|v|/2}|u-v|^\gamma e^{-|u|^2/16}e^{-|v|^2/64}w^2(\ell,\vartheta)(v)|\partial^{\alpha_1}_{\beta_1}g_1(u')\partial^{\alpha_2}_{\beta_2}\\&\quad\times g_2(v')\partial^\alpha_\beta g_3(v)|d\omega dudvdx\\&\leqq C\left\{\int\int|u-v|^\gamma e^{-|u|^2/16}e^{-|v|^2/64}w^2|\partial^{\alpha_1}_{\beta_1}g_1(u')|^2|\partial^{\alpha_2}_{\beta_2}g_2(v')|^2d\omega dudvdx\right\}^{1/2}\\&\quad\times\left\{\int\int|u-v|^\gamma e^{-|u|^2/16}e^{-|v|^2/64}w^2|\partial^\alpha_\beta g_3(v)|^2dudvdx\right\}^{1/2}\\&\leqq C\left\{\int\int|u'-v'|^\gamma e^{-\frac{1}{64}(|u'|^2+|v'|^2)}w^2(v)|\partial^{\alpha_1}_{\beta_1}g_1(u')|^2|\partial^{\alpha_2}_{\beta_2}g_2(v')|^2\right\}^{1/2}\\&\quad\times||\partial^\alpha_\beta g_3||_{\nu,\ell,\vartheta}.\end{aligned}$$

In the first factor we have used (4) and $|u'-v'|=|u-v|$. By (4),

$$e^{\frac{q}{4}(1+|v|^2)^{\vartheta/2}}\leqq e^{\frac{q}{4}(1+|v'|^2+|u'|^2)^{\vartheta/2}}\leqq e^{\frac{q}{4}(1+|v'|^2)^{\vartheta/2}}e^{\frac{q}{4}(1+|u'|^2)^{\vartheta/2}}. \tag{39}$$

Using this, (10) and (4) we have

$$e^{-\frac{1}{64}(|u'|^2+|v'|^2)}w^2(\ell,\vartheta)(v)\leqq e^{-\frac{1}{128}(|u'|^2+|v'|^2)}w^2(\ell,\vartheta)(v')w^2(\ell,\vartheta)(u').$$

So that the factor in braces is

$$\leqq C\int|u'-v'|^\gamma e^{-\frac{1}{128}(|u'|^2+|v'|^2)}w^2(u')|\partial^{\alpha_1}_{\beta_1}g_1(u')|^2w^2(v')|\partial^{\alpha_2}_{\beta_2}g_2(v')|^2d\omega dudvdx.$$

The change of variables $(u,v)\to(u',v')$ implies

$$=C\left\{\int\int|u-v|^\gamma e^{-\frac{1}{128}(|u|^2+|v|^2)}w^2(u)|\partial^{\alpha_1}_{\beta_1}g_1(u)|^2w^2(v)|\partial^{\alpha_2}_{\beta_2}g_2(v)|^2dudvdx\right\}^{1/2}.$$

Assume $|\alpha_1|+|\beta_1|\leqq N/2$. As in (37),

$$\sup_{x,u}\left\{w(\ell,\vartheta)(u)e^{-\frac{1}{256}|u|^2}|\partial^{\alpha_1}_{\beta_1}g_1(x,u)|\right\}\leqq C\mathcal{E}^{1/2}_{l,\vartheta}(g_1).$$

Integrate first over du to obtain

$$\begin{aligned}\int|u-v|^\gamma e^{-\frac{1}{128}|u|^2}w(\ell,\vartheta)(u)|\partial^{\alpha_1}_{\beta_1}g_1(u)|^2du&\leqq C\mathcal{E}^{1/2}_{l,\vartheta}(g_1)\int|u-v|^\gamma e^{-\frac{1}{256}|u|^2}du\\&\leqq C\mathcal{E}^{1/2}_{l,\vartheta}(g_1)[1+|v|]^\gamma.\end{aligned}$$

If $|\alpha_2|+|\beta_2|\leqq N/2$ use this last argument but switch $\partial^{\alpha_1}_{\beta_1}g_1$ with $\partial^{\alpha_2}_{\beta_2}g_2$.

Then the bound for the gain term over $\{|u|\geqq|v|/2\}$, if $|\alpha_1|+|\beta_1|\leqq N/2$, is

$$\int_{|u|\geqq|v|/2}\leqq C\mathcal{E}^{1/2}_{l,\vartheta}(g_1)||\partial^{\alpha_2}_{\beta_2}g_2||_{\nu,\ell,\vartheta}||\partial^\alpha_\beta g_3||_{\nu,\ell,\vartheta}.$$

And the bound for the gain term over $\{|u| \geqq |v|/2\}$, if $|\alpha_2| + |\beta_2| \leqq N/2$, is

$$\int_{|u|\geqq|v|/2} \leqq C||\partial^{\alpha_2}_{\beta_2} g_1||_{\nu,\ell,\vartheta}\mathcal{E}^{1/2}_{l,\vartheta}(g_2)||\partial^{\alpha}_{\beta} g_3||_{\nu,\ell,\vartheta}.$$

This completes the estimate for the gain term over $\{|u| \geqq |v|/2\}$.

Case (2b): *The Gain Term over* $\{|u| \leqq |v|/2, |v| \leqq 1\}$.

Now consider $\{|u| \leqq |v|/2\}$. Since $|v-u| < |v|/2$ implies $|u| \geqq |v|-|v-u| > |v|/2$, we obtain

$$\{|u| \leqq |v|/2\} = \{|u| \leqq |v|/2\} \cap \{|v-u| \geqq |v|/2\}.$$

Since $|v| \leqq 1$, then $|u| \leqq 1/2$ and the gain term is bounded by

$$\begin{aligned}
&\int_{|v|\leqq 1, |u|\leqq|v|/2} |u-v|^{\gamma} e^{-|u|^2/8} w^2 |\partial^{\alpha_1}_{\beta_1} g_1(u')\partial^{\alpha_2}_{\beta_2} g_2(v')\partial^{\alpha}_{\beta} g_3(v)| d\omega du dv dx \\
&\leqq C\int_{|v|\leqq 1}\left\{|v|^{\gamma/2}\int_{|u|\leqq 1/2}|u-v|^{\gamma/2}e^{-|u|^2/8}|\partial^{\alpha_1}_{\beta_1} g_1(u')\partial^{\alpha_2}_{\beta_2} g_2(v')|d\omega du\right\} \\
&\quad \times|\partial^{\alpha}_{\beta} g_3(v)|dvdx \\
&\quad \times \leqq C\int_{|v|\leqq 1}\left\{|v|^{\gamma}\int_{|u|\leqq 1/2}|\partial^{\alpha_1}_{\beta_1} g_1(u')|^2|\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 d\omega du\right\}^{1/2} \\
&\quad \times\left\{\int_{|u|\leqq 1/2}|u-v|^{\gamma}e^{-|u|^2/4}du\right\}^{1/2}|\partial^{\alpha}_{\beta} g_3(v)|dvdx \\
&\leqq C\int_{|v|\leqq 1}\left\{|v|^{\gamma}\int_{|u|\leqq 1/2}|\partial^{\alpha_1}_{\beta_1} g_1(u')|^2|\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 d\omega du\right\}^{1/2}|\partial^{\alpha}_{\beta} g_3(v)|dvdx \\
&\leqq C\left\{\int_{|v|\leqq 1,|u|\leqq 1/2}|v|^{\gamma}|\partial^{\alpha_1}_{\beta_1} g_1(u')|^2|\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 d\omega du dv dx\right\}^{1/2}\|\partial^{\alpha}_{\beta} g_3\|_{\nu,\ell}.
\end{aligned} \tag{40}$$

We now estimate the first factor. Since $|u| \leqq |v|/2$, from (3) we have

$$|u'| + |v'| \leqq 2[|u| + |v|] \leqq 3|v|.$$

Since $\gamma < 0$, this implies

$$|v|^{\gamma} \leqq 3^{-\gamma}|u'|^{\gamma}, \quad |v|^{\gamma} \leqq 3^{-\gamma}|v'|^{\gamma}.$$

Thus,

$$\begin{aligned}
&\int_{|v|\leqq 1,|u|\leqq 1/2}|v|^{\gamma}|\partial^{\alpha_1}_{\beta_1} g_1(u')|^2|\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 d\omega du dv \\
&\leqq C\int_{|v'|\leqq 3,|u'|\leqq 3}\min[|v'|^{\gamma}, |u'|^{\gamma}]|\partial^{\alpha_1}_{\beta_1} g_1(u')|^2|\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 d\omega du dv dx.
\end{aligned}$$

Now change variables $(v, u) \to (v', u')$ so that the above is

$$C \int_{|v| \leqq 3, |u| \leqq 3} \min[|v|^\gamma, |u|^\gamma] |\partial_{\beta_1}^{\alpha_1} g_1(u)|^2 |\partial_{\beta_2}^{\alpha_2} g_2(v)|^2 d\omega du dv dx.$$

Assume $|\alpha_1| + |\beta_1| \leqq N/2$ and majorize the above by

$$\begin{aligned} &C \int \left\{ \int_{|u| \leqq 3} |u|^\gamma |\partial_{\beta_1}^{\alpha_1} g_1(u)|^2 du \right\} \left\{ \int_{|v| \leqq 3} |\partial_{\beta_2}^{\alpha_2} g_2(v)|^2 dv \right\} dx \\ &\leqq C \sup_{x, |u| \leqq 3} |\partial_{\beta_1}^{\alpha_1} g_1(u)|^2 \|\partial_{\beta_2}^{\alpha_2} g_2\|_{\boldsymbol{\nu}, \ell}^2 \leqq C \mathcal{E}_l(g_1) \|\partial_{\beta_2}^{\alpha_2} g_2\|_{\boldsymbol{\nu}, \ell}^2. \end{aligned}$$

Alternatively, if $|\alpha_2| + |\beta_2| \leqq N/2$ then

$$\begin{aligned} &C \int \left\{ \int_{|u| \leqq 3} |\partial_{\beta_1}^{\alpha_1} g_1(u)|^2 du \right\} \left\{ \int_{|v| \leqq 3} |v|^\gamma |\partial_{\beta_2}^{\alpha_2} g_2(v)|^2 dv \right\} dx \\ &\leqq C \|\partial_{\beta_1}^{\alpha_1} g_1\|_{\boldsymbol{\nu}, \ell}^2 \sup_{x, |v| \leqq 3} |\partial_{\beta_2}^{\alpha_2} g_2(v)|^2 \leqq C \|\partial_{\beta_1}^{\alpha_1} g_1\|_{\boldsymbol{\nu}, \ell}^2 \mathcal{E}_l(g_2). \end{aligned}$$

Combine this upper bound with (40) to complete the estimate for the gain term over $\{|u| \leqq |v|/2, \ \ |v| \leqq 1\}$.

Case (2c): *The Gain Term over* $\{|u| \leqq |v|/2, \ \ |v-u| \geqq |v|/2, |v| \geqq 1\}$.

The last case is the gain term over the region $\{|u| \leqq |v|/2, \ \ |v-u| \geqq |v|/2, |v| \geqq 1\}$. The integral of $w^2(\ell, \vartheta) \Gamma_{\text{gain}}^0 [\partial_{\beta_1}^{\alpha_1} g_1, \partial_{\beta_2}^{\alpha_2} g_2] \partial_\beta^\alpha g_3$ over such a region is bounded by

$$\begin{aligned} &\int_{|u| \leqq |v|/2, |v| \geqq 1} |u - v|^\gamma e^{-|u|^2/4} w^2(\ell, \vartheta) |\partial_{\beta_1}^{\alpha_1} g_1(u') \partial_{\beta_2}^{\alpha_2} g_2(v') \partial_\beta^\alpha g_3(v)| d\omega du dv dx \\ &\leqq C \int_{|u| \leqq |v|/2, |v| \geqq 1} [1 + |v|]^\gamma e^{-|u|^2/4} w^2(\ell, \vartheta) |\partial_{\beta_1}^{\alpha_1} g_1(u') \partial_{\beta_2}^{\alpha_2} g_2(v') \\ &\quad \times \partial_\beta^\alpha g_3(v)| d\omega du dv dx \\ &\leqq C \left\{ \int [1 + |v|]^\gamma e^{-|u|^2/4} w^2(\ell, \vartheta) |\partial_{\beta_1}^{\alpha_1} g_1(u')|^2 |\partial_{\beta_2}^{\alpha_2} g_2(v')|^2 d\omega du dv dx \right\}^{1/2} \\ &\quad \times \left\{ \int [1 + |v|]^\gamma e^{-|u|^2/4} w^2(\ell, \vartheta) |\partial_\beta^\alpha g_3(v)|^2 d\omega du dv dx \right\}^{1/2} \qquad (41) \\ &\leqq C \left\{ \int [1 + |v|]^\gamma w^2(\ell, \vartheta)(v) |\partial_{\beta_1}^{\alpha_1} g_1(u')|^2 |\partial_{\beta_2}^{\alpha_2} g_2(v')|^2 d\omega du dv dx \right\}^{1/2} \\ &\quad \times ||\partial_\beta^\alpha g_3||_{\boldsymbol{\nu}, \ell, \vartheta}. \end{aligned}$$

We have used $|v - u|^\gamma \leqq 4^{-\gamma}[1 + |v|]^\gamma$ in the first inequality. If $\ell\tau < 0$, use $|v| \geqq 2|u|$ and (4) to establish

$$(1 + |v|^2)^{\ell\tau/2} \leqq C(1 + |v'|^2 + |u'|^2)^{\ell\tau/2}.$$

Conversely, if $\ell\tau \geqq 0$, just use (4) to establish the same inequality. Recall (39) and $M(v) = \exp\left(\frac{q}{4}(1+|v|^2)^{\vartheta/2}\right)$. Thus,

$$w^2(\ell,\vartheta)(v) \leqq C(1+|v'|^2+|u'|^2)^{\ell\tau} M(v')M(u').$$

Then, since $\ell = |\beta| - l$ and $\tau \leqq 0$, $l \geqq 0$, we have

$$\begin{aligned} w^2(\ell,\vartheta)(v) &\leqq C(1+|v'|^2+|u'|^2)^{-l\tau}(1+|v'|^2+|u'|^2)^{|\beta|\tau} M(v')M(u') \\ &\leqq C(1+|v'|^2)^{-l\tau}(1+|u'|^2)^{-l\tau}(1+|v'|^2+|u'|^2)^{|\beta|\tau} M(v')M(u'). \end{aligned}$$

Assume $|\alpha_2|+|\beta_2| \leqq N/2$. Using this estimate and the change of variable $(v,u) \to (v',u')$ we obtain

$$\begin{aligned} &\int [1+|v|]^\gamma w^2(\ell,\vartheta)(v)|\partial^{\alpha_1}_{\beta_1} g_1(u')|^2 |\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 d\omega du dv dx \\ &\leqq C\int [1+|u'|]^\gamma w^2(|\beta_1|-l,\vartheta)(u')|\partial^{\alpha_1}_{\beta_1} g_1(u')|^2 w^2(|\beta_2|-l,\vartheta)(v')|\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 \\ &= C\int [1+|u|]^\gamma w^2(|\beta_1|-l,\vartheta)(u)|\partial^{\alpha_1}_{\beta_1} g_1(u)|^2 w^2(|\beta_2|-l,\vartheta)(v) \\ &\quad \times |\partial^{\alpha_2}_{\beta_2} g_2(v)|^2 du dv dx \\ &= C\int \left\{\int w^2(|\beta_2|-l,\vartheta)(v)|\partial^{\alpha_2}_{\beta_2} g_2(v)|^2 dv\right\} \\ &\quad \times \left\{\int [1+|u|]^\gamma w^2(|\beta_1|-l,\vartheta)(u)|\partial^{\alpha_1}_{\beta_1} g_1(u)|^2 du\right\} dx. \end{aligned}$$

Using the embedding in (38), we see that this is bounded by

$$\begin{aligned} C\mathcal{E}_{l,\vartheta}(g_2) &\int [1+|u|]^\gamma w^2(|\beta_1|-l,\vartheta)(u)|\partial^{\alpha_1}_{\beta_1} g_1(u)|^2 du dx \\ &\leqq C\mathcal{E}_{l,\vartheta}(g_2)||\partial^{\alpha_1}_{\beta_1} g_1||^2_{\boldsymbol{\nu},|\beta_1|-l,\vartheta}. \end{aligned}$$

Similarly if $|\alpha_1|+|\beta_1| \leqq N/2$ then this is bounded by

$$C||\partial^{\alpha_2}_{\beta_2} g_2||^2_{\boldsymbol{\nu},|\beta_2|-l,\vartheta}\mathcal{E}_{l,\vartheta}(g_1).$$

Combine this with (41) to complete the nonlinear estimate. □

This completes the estimates for the Boltzmann case. In Section 4 we use these to establish global existence. Then in Section 5 we prove the decay. In the next section we establish the analogous estimates for the Landau case.

3. Landau Estimates

In this section, we will prove the basic estimates used to obtain global existence of solutions with an exponential weight in the Landau case. In this case, the derivatives in the Landau operator cause extra difficulties in particular because $\partial_i w(\ell, \vartheta)$ can grow faster in v than $w(\ell, \vartheta)$. This new feature of the exponential weight (10) forces us to weaken the linear estimate with high order velocity derivatives (Lemma 8) from the analogous estimate [9, Lemma 6, p.403]. A new linear estimate with no extra derivatives (Lemma 9) is also necessary because of the exponential weight. It turns out that we need to dig up exact cancellation in order to prove this estimate in the $\vartheta = 2$ case.

For any vector-valued function $\mathbf{g}(v) = (g_1, g_2, g_3)$, we define the projection to the vector v as

$$P_v g_i \equiv \frac{v_i}{|v|} \sum_{j=1}^{3} \frac{v_j}{|v|} g_j. \tag{42}$$

Furthermore, in this section we will use the Einstein summation convention over i and j, for example repeated indices are always summed

$$\sigma^i(v) = \sigma^{ij}(v) \frac{v_j}{2} = \sum_{j=1}^{3} \sigma^{ij}(v) \frac{v_j}{2}. \tag{43}$$

With this notation we have

Lemma 4. [4, 9]$\sigma^{ij}(v)$, $\sigma^i(v)$ *are smooth functions such that*

$$|\partial_\beta \sigma^{ij}(v)| + |\partial_\beta \sigma^i(v)| \leqq C_\beta [1 + |v|]^{\gamma + 2 - |\beta|}$$

and furthermore

$$\sigma^{ij}(v) = \lambda_1(v) \frac{v_i v_j}{|v|^2} + \lambda_2(v) \left(\delta_{ij} - \frac{v_i v_j}{|v|^2} \right). \tag{44}$$

Thus

$$\sigma^{ij}(v) g_i g_j = \lambda_1(v) \sum_{i=1}^{3} \{P_v g_i\}^2 + \lambda_2(v) \sum_{i=1}^{3} \{[I - P_v] g_i\}^2. \tag{45}$$

Moreover, there are constants c_1 and $c_2 > 0$ such that as $|v| \to \infty$

$$\lambda_1(v) \sim c_1 [1 + |v|]^{\gamma}, \quad \lambda_2(v) \sim c_2 [1 + |v|]^{\gamma + 2}.$$

The estimate of σ^{ij} and σ^i with high derivatives was already established in [4]. The computation of the eigenvalues and their convergence rate was already shown in [9]. We prove the representation (44) below because we will use it in important places in later proofs and it is not formally written down in the other papers.

Proof. Recall from (9) that

$$\sigma^{ij}(v) = (2\pi)^{-3/2}\int_{\mathbb{R}^3}\left(\delta_{ij} - \frac{(v_i - u_i)(v_j - u_j)}{|v-u|^2}\right)|v-u|^{\gamma+2}e^{-|u|^2/2}du.$$

Changing variables $u \to v - u$ we have

$$\sigma^{ij}(v) = (2\pi)^{-3/2}\int_{\mathbb{R}^3}\left(\delta_{ij} - \frac{u_i u_j}{|u|^2}\right)|u|^{\gamma+2}e^{-|v-u|^2/2}du.$$

Given $v \in \mathbb{R}^3$ define $v^1 = v/|v|$ and complete an orthonormal basis $\{v^1, v^2, v^3\}$ where $v^i \cdot v^j = \delta_{ij}$. Then define the corresponding orthogonal 3×3 matrix as

$$\mathcal{O} = [v^1\ v^2\ v^3].$$

Applying this orthogonal transformation to the integral in σ^{ij} above we obtain

$$\sigma^{ij}(v) = (2\pi)^{-3/2}\int_{\mathbb{R}^3}\left(\delta_{ij} - \frac{(\mathcal{O}u)_i(\mathcal{O}u)_j}{|u|^2}\right)|u|^{\gamma+2}e^{-|v-\mathcal{O}u|^2/2}du.$$

Here we have used $|\mathcal{O}u| = |u|$. Also

$$(\mathcal{O}u)_i = u_1 v_i^1 + u_2 v_i^2 + u_3 v_i^3$$

and

$$|v - \mathcal{O}u|^2 = |v - u_1 v^1 - u_2 v^2 - u_3 v^3|^2 = (|v| - u_1)^2 + u_2^2 + u_3^2. \tag{46}$$

By symmetry, we have

$$\sigma^{ij}(v) = (2\pi)^{-3/2}\int_{\mathbb{R}^3}\left(\delta_{ij} - \sum_{l,m=1}^{3}\frac{u_l u_m}{|u|^2}v_i^l v_j^m\right)|u|^{\gamma+2}e^{-\frac{1}{2}[(|v|-u_1)^2+u_2^2+u_3^2]}du$$
$$= (2\pi)^{-3/2}\int_{\mathbb{R}^3}\left(\delta_{ij} - \sum_{m=1}^{3}\frac{u_m^2}{|u|^2}v_i^m v_j^m\right)|u|^{\gamma+2}e^{-\frac{1}{2}[(|v|-u_1)^2+u_2^2+u_3^2]}du.$$

Write $(m = 1, 2, 3)$

$$B_0(v) \equiv (2\pi)^{-3/2}\int_{\mathbb{R}^3}|u|^{\gamma+2}e^{-\frac{1}{2}[(|v|-u_1)^2+u_2^2+u_3^2]}du,$$
$$B_m(v) \equiv (2\pi)^{-3/2}\int_{\mathbb{R}^3}\frac{u_m^2}{|u|^2}|u|^{\gamma+2}e^{-\frac{1}{2}[(|v|-u_1)^2+u_2^2+u_3^2]}du.$$

Then by symmetry

$$B_2(v) = B_3(v) = (2\pi)^{-3/2}\int_{\mathbb{R}^3}\frac{u_2^2 + u_3^2}{2|u|^2}|u|^{\gamma+2}e^{-\frac{1}{2}[(|v|-u_1)^2+u_2^2+u_3^2]}du$$

and we have

$$\sigma^{ij}(v) = B_0(v)\delta_{ij} - B_1(v)v_j^1 v_j^1 - B_2(v)(v_i^2 v_j^2 + v_i^3 v_j^3).$$

Define the orthogonal projections $P_j = v^j \otimes v^j$. Then we have the resolution of the identity $I = P_1 + P_2 + P_3$. In component form this is

$$v_i^2 v_j^2 + v_i^3 v_j^3 = \delta_{ij} - \frac{v_i v_j}{|v|^2}.$$

Then we have

$$\sigma^{ij}(v) = \{B_0(v) - B_1(v)\} \frac{v_i v_j}{|v|^2} + \{B_0(v) - B_2(v)\} \left(\delta_{ij} - \frac{v_i v_j}{|v|^2} \right).$$

Now the eigenvalues are $\lambda_1(v) = B_0(v) - B_1(v)$ and $\lambda_2(v) = B_0(v) - B_2(v)$ or

$$\lambda_1(v) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} |u|^{\gamma+2} \left(1 - \frac{u_1^2}{|u|^2} \right) e^{-\frac{1}{2}[(|v|-u_1)^2 + u_2^2 + u_3^2]} du$$

and

$$\lambda_2(v) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} |u|^{\gamma+2} \left(1 - \frac{u_2^2 + u_3^2}{2|u|^2} \right) e^{-\frac{1}{2}[(|v|-u_1)^2 + u_2^2 + u_3^2]} du.$$

This completes the derivation of the spectral representation for $\sigma^{ij}(v)$. $\square$

Next, we write bounds for the σ norm.

Lemma 5. *[9,* Corollary 1, p.399*] There exists* $c > 0$ *such that*

$$c|g|^2_{\sigma,\ell,\vartheta} \geqq \left| [1+|v|]^{\frac{\gamma}{2}} \{P_v \partial_i g\} \right|^2_{\ell,\vartheta} + \left| [1+|v|]^{\frac{\gamma+2}{2}} \{[I - P_v]\partial_i g\} \right|^2_{\ell,\vartheta} + \left| [1+|v|]^{\frac{\gamma+2}{2}} g \right|^2_{\ell,\vartheta}.$$

Furthermore,

$$\frac{1}{c}|g|^2_{\sigma,\ell,\vartheta} \leqq \left| [1+|v|]^{\frac{\gamma}{2}} \{P_v \partial_i g\} \right|^2_{\ell,\vartheta} + \left| [1+|v|]^{\frac{\gamma+2}{2}} \{[I - P_v]\partial_i g\} \right|^2_{\ell,\vartheta} + \left| [1+|v|]^{\frac{\gamma+2}{2}} g \right|^2_{\ell,\vartheta}.$$

The upper bound was not written down in [9], but the proof is the same. We write it down here because we will use it in the nonlinear estimate. Next, the operators A, K and Γ from (5) and (8) in the Landau case are defined.

Lemma 6. *[9,* Lemma 1*] We have the following representations for* A, K *and* Γ.

$$Ag_2 = \partial_i[\sigma^{ij}\partial_j g_2] - \sigma^{ij} \frac{v_i}{2} \frac{v_j}{2} g_2 + \partial_i \sigma^i g_2 \tag{47}$$

$$Kg_1 = -\mu^{-1/2}\partial_i \left\{ \mu \left[\phi^{ij} * \left\{ \mu^{1/2} \left[\partial_j g_1 + \frac{v_j}{2} g_1 \right] \right\} \right] \right\} \tag{48}$$

$$= -\mu^{-1/2}\partial_i \left\{ \mu \int_{\mathbb{R}^3} \phi^{ij}(v - v')\mu^{1/2}(v') \left[\partial_j g_1(v') + \frac{v'_j}{2} g_1(v') \right] dv' \right\},$$

$$\Gamma[g_1, g_2] = \partial_i[\{\phi^{ij} * [\mu^{1/2} g_1]\}\partial_j g_2] - \left\{ \phi^{ij} * \left[\frac{v_i}{2} \mu^{1/2} g_1 \right] \right\} \partial_j g_2 - \partial_i[\{\phi^{ij} * [\mu^{1/2}\partial_j g_1]\} g_2] + \left\{ \phi^{ij} * \left[\frac{v_i}{2} \mu^{1/2} \partial_j g_1 \right] \right\} g_2. \tag{49}$$

These representations are different in a few places by a factor of $\frac{1}{2}$ from those in [9]. The only reason for this difference is our use of a different normalization for the Maxwellian in this paper.

Proof. We only reprove A. First notice that for either fixed i or j

$$\sum_i \phi^{ij}(v)v_i = \sum_j \phi^{ij}(v)v_j = 0. \tag{50}$$

We now take the derivatives inside Ag_2

$$
\begin{aligned}
Ag_2 &= \mu^{-1/2}Q(\mu, \mu^{1/2}g_2) \\
&= \mu^{-1/2}\partial_i \left\{\sigma^{ij}\mu^{1/2}\left[\partial_j g_2 - \frac{v_j}{2}g_2\right]\right\} + \mu^{-1/2}\partial_i\{\{\phi^{ij} * [v_j\mu]\}\mu^{1/2}g_2\} \\
&= \mu^{-1/2}\partial_i \left\{\sigma^{ij}\mu^{1/2}\left[\partial_j g_2 - \frac{v_j}{2}g_2\right]\right\} \\
&\quad +\mu^{-1/2}\partial_i\{[\phi^{ij} * \mu]v_j\mu^{1/2}g_2\} \qquad \text{by (50)} \\
&= \mu^{-1/2}\partial_i \left\{\sigma^{ij}\mu^{1/2}\left[\partial_j g_2 + \frac{v_j}{2}g_2\right]\right\} \\
&= \mu^{-1/2}\partial_i\{\sigma^{ij}\mu^{1/2}\partial_j g_2\} + \mu^{-1/2}\partial_i \left\{\sigma^{ij}\mu^{1/2}\frac{v_j}{2}g_2\right\} \\
&= \partial_i[\sigma^{ij}\partial_j g_2] + \mu^{-1/2}\partial_i[\mu^{1/2}]\sigma^{ij}\partial_j g_2 + \partial_i \left\{\sigma^{ij}\frac{v_j}{2}g_2\right\} \\
&\quad +\mu^{-1/2}\partial_i[\mu^{1/2}]\sigma^{ij}\frac{v_j}{2}g_2 \\
&= \partial_i[\sigma^{ij}\partial_j g_2] - \sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g_2 + \mu^{-1/2}\partial_i[\mu^{1/2}]\sigma^{ij}\partial_j g_2 + \partial_i \left\{\sigma^{ij}\frac{v_j}{2}g_2\right\} \\
&= \partial_i[\sigma^{ij}\partial_j g_2] - \sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g_2 + \partial_i \left\{\sigma^{ij}\frac{v_j}{2}\right\} g_2,
\end{aligned}
$$

from (9), where $\partial_i[\mu^{1/2}] = \frac{v_i}{2}\mu^{1/2}$. □

In the rest of this section, we will prove estimates for A, K and Γ.

Lemma 7. *Let* $0 \leqq \vartheta \leqq 2$, $\ell \in \mathbb{R}$ *and* $0 < q$. *If* $\vartheta = 2$ *restrict* $0 < q < 1$. *Then for any* $\eta > 0$, *there is* $0 < C = C(\eta) < \infty$ *such that*

$$|\langle w^2\partial_i\sigma^i g_1, g_2\rangle| + |\langle w^2 K g_1, g_2\rangle| \leqq \eta\, |g_1|_{\sigma,\ell,\vartheta}\, |g_2|_{\sigma,\ell,\vartheta} + C|g_1\bar{\chi}_C|_\ell |g_2\bar{\chi}_C|_\ell, \tag{51}$$

where $w^2 = w^2(\ell, \vartheta)$ *and* $\bar{\chi}_{C(\eta)}$ *is defined in* (21).

The estimate for the $\partial_i\sigma^i$ term is exactly the same as in [9]. But, as in the Boltzmann case, the estimate for K needs modification. So we need to show that K can control one exponentially growing factor $w(\ell, \vartheta)(v)$. We remark that, although it is not used in this paper, the proof clearly shows $|\langle w^2 K g_1, g_2\rangle| \leqq \eta\, |g_1|_{\sigma,\ell}\, |g_2|_{\sigma,\ell,\vartheta} + C|g_1\bar{\chi}_C|_\ell |g_2\bar{\chi}_C|_\ell$.

Proof. For $m > 0$, we split

$$\int w^2 \partial_i \sigma^i g_1 g_2 = \int_{\{|v|\leqq m\}} + \int_{\{|v|\geqq m\}}.$$

By Lemma 4, $|\partial_i \sigma^i| \leqq C[1+|v|]^{\gamma+1}$. Thus, the integral over $\{|v| \leqq m\}$ is $\leqq C(m)|g_1 \bar{\chi}_m|_\ell |g_2 \bar{\chi}_m|_\ell$. From Lemma 5 and the Cauchy–Schwartz inequality

$$\int_{\{|v|\geqq m\}} w^2 |\partial_i \sigma^i g_1 g_2| dv \leqq \frac{C}{m} \int w^2 [1+|v|]^{\gamma+2} |g_1 g_2| \leqq \frac{C}{m} |g_1|_{\sigma,\ell,\vartheta} |g_2|_{\sigma,\ell,\vartheta}. \tag{52}$$

This completes (51) for the $\partial_i \sigma^i$ term.

Recalling the linear operator K in (48), we have

$$\begin{aligned} w^2 K g_1 = &-\partial_i \left\{ w^2 \mu^{1/2} \left[\phi^{ij} * \left\{ \mu^{1/2} \partial_j g_1 + \frac{v_j}{2} \mu^{1/2} g_1 \right\} \right] \right\} \\ &+\partial_i (w^2) \mu^{1/2} \left[\phi^{ij} * \left\{ \mu^{1/2} \partial_j g_1 + \frac{v_j}{2} \mu^{1/2} g_1 \right\} \right] \\ &+ w^2 \frac{v_i}{2} \mu^{1/2} \left[\phi^{ij} * \left\{ \mu^{1/2} \partial_j g_1 + \frac{v_j}{2} \mu^{1/2} g_1 \right\} \right]. \end{aligned} \tag{53}$$

The derivative of the weight function is

$$\partial_i (w^2(\ell, \vartheta)) = w^2(\ell, \vartheta) w_1(v) v_i, \tag{54}$$

where

$$w_1(v) = \left\{ 2\ell\tau(1+|v|^2)^{-1} + q\frac{\vartheta}{2}(1+|v|^2)^{\frac{\vartheta}{2}-1} \right\}. \tag{55}$$

After integrating by parts for the first term and collecting terms, we can rewrite $\langle w^2 K g_1, g_2 \rangle$ as

$$\sum_{|\beta_1|,|\beta_2|\leqq 1} \int w^2(v) \phi^{ij}(v-v') \mu^{1/2}(v) \mu^{1/2}(v') \bar{\mu}_{\beta_1\beta_2}(v,v') \partial_{\beta_1} g_1(v') \partial_{\beta_2} g_2(v) dv' dv,$$

where $\bar{\mu}_{\beta_1\beta_2}(v,v')$ is a collection of smooth functions satisfying

$$|\nabla_v \bar{\mu}_{\beta_1\beta_2}(v,v')| + |\nabla_{v'} \bar{\mu}_{\beta_1\beta_2}(v,v')| + |\bar{\mu}_{\beta_1\beta_2}(v,v')| \leqq C(1+|v'|^2)^{1/2}(1+|v|^2)^{1/2}.$$

Since either $0 \leqq \vartheta < 2$ or $\vartheta = 2$ and $0 < q < 1$, there exists $0 < q' < 1$ such that

$$w(\ell, \vartheta)(v) \mu^{1/2}(v) \leqq C \mu^{q'/2}(v). \tag{56}$$

In fact, if $0 \leqq \vartheta < 2$ choose any $0 < q' < 1$ and if $\vartheta = 2$ choose $0 < q' < 1 - q$. Therefore, we can rewrite $\langle w^2 K g_1, g_2 \rangle$ as

$$\sum_{|\beta_1|,|\beta_2|\leqq 1} \int w(v) \phi^{ij}(v-v') \mu^{q'/4}(v) \mu^{1/4}(v') \mu_{\beta_1\beta_2}(v,v') \partial_{\beta_1} g_1(v') \partial_{\beta_2} g_2(v) dv' dv,$$

where $\mu_{\beta_1\beta_2}(v,v')$ is a different collection of smooth functions satisfying

$$|\nabla_v \mu_{\beta_1\beta_2}(v,v')| + |\nabla_{v'} \mu_{\beta_1\beta_2}(v,v')| + |\mu_{\beta_1\beta_2}(v,v')| \leqq C e^{-\frac{q'}{16}|v|^2} e^{-\frac{1}{16}|v'|^2}.$$

We have removed an exponentially growing factor $w(\ell, \vartheta)(v)$.

Since $\phi^{ij}(v) = O(|v|^{\gamma+2}) \in L^2_{loc}(\mathbb{R}^3)$ and $\gamma \geqq -3$, Fubini's Theorem implies

$$\phi^{ij}(v - v')\mu^{q'/4}(v)\mu^{1/4}(v') \in L^2(\mathbb{R}^3 \times \mathbb{R}^3).$$

Therefore, for any given $m > 0$, we can choose a C_c^∞ function $\psi^{ij}(v, v')$ such that

$$||\phi^{ij}(v - v')\mu^{q'/4}(v)\mu^{1/4}(v') - \psi^{ij}(v, v')||_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \leqq \frac{1}{m},$$
$$\text{supp}\{\psi^{ij}\} \subset \{|v'| + |v| \leqq C(m)\} < \infty.$$

We split

$$\phi^{ij}(v - v')\mu^{q'/4}(v)\mu^{1/4}(v') = \psi^{ij} + [\phi^{ij}(v - v')\mu^{q'/4}(v)\mu^{1/4}(v') - \psi^{ij}].$$

Then

$$\langle w^2 K g_1, g_2\rangle = J_1[g_1, g_2] + J_2[g_1, g_2], \tag{57}$$

where

$$J_1 = \int w(v)\psi^{ij}(v, v')\mu_{\beta_1\beta_2}(v, v')\partial_{\beta_1} g_1(v')\partial_{\beta_2} g_2(v)dv'dv,$$
$$J_2 = \int w(v)[\phi^{ij}(v - v')\mu^{q'/4}(v)\mu^{1/4}(v') - \psi^{ij}]\mu_{\beta_1\beta_2}(v, v')\partial_{\beta_1} g_1(v')\partial_{\beta_2} g_2(v)dv'dv.$$

Above we are implicitly summing over $|\beta_1|, |\beta_2| \leqq 1$. We will bound each of these terms separately.

The J_2 term is bounded as

$$\begin{aligned}|J_2| &\leqq ||\phi^{ij}(v - v')\mu^{q'/4}(v)\mu^{1/4}(v') - \psi^{ij}(v, v')||_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \\ &\quad\times ||w(v)\mu_{\beta_1\beta_2}(v, v')\partial_{\beta_1} g_1(v')\partial_{\beta_2} g_2(v)||_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \\ &\leqq \frac{C}{m}\left|\mu^{1/16}\partial_{\beta_1} g_1\right|_0 \left|\mu^{q'/16}\partial_{\beta_2} g_2\right|_{\ell,\vartheta} \leqq \frac{C}{m}|g_1|_{\sigma,\ell}\,|g_2|_{\sigma,\ell,\vartheta}\,.\end{aligned}$$

Now for the first term J_1, integrations by parts over v and v' variables yields

$$\begin{aligned}|J_1| &= \left|(-1)^{\beta_1+\beta_2}\int \partial_{\beta_2}[w(v)\partial_{\beta_1}\{\psi^{ij}(v, v')\mu_{\beta_1\beta_2}(v, v')\}]g_1(v')g_2(v)\right| \\ &\leqq C||\psi^{ij}||_{C^2}\left\{\int_{|v|\leqq C(m)} |g_1|^2 dv\right\}^{1/2}\left\{\int_{|v|\leqq C(m)} |g_2|^2 dv\right\}^{1/2}.\end{aligned}$$

We thus conclude (51) by choosing $m > 0$ large enough.

Next, we estimate the linear terms with velocity derivatives.

Lemma 8. *Let $|\beta| > 0$, $\ell \in \mathbb{R}$, $0 \leqq \vartheta \leqq 2$ and $q > 0$. If $\vartheta = 2$ fix $0 < q < 1$. Then for any small $\eta > 0$, there exists $C(\eta) > 0$ such that*

$$|\langle w^2(\ell,\vartheta)\partial_\beta[Kg_1], g_2\rangle| \leqq \left\{\eta \sum_{|\bar\beta|\leqq|\beta|} \left|\partial_{\bar\beta} g_1\right|_{\sigma,\ell} + C(\eta)\left|\bar\chi_{C(\eta)} g_1\right|_\ell\right\} |g_2|_{\sigma,\ell,\vartheta} \cdot$$

Further if $\tau \leqq -1$ in (10) and $\ell = r - l$, where $l \geqq 0$ and $r \geqq |\beta|$, then

$$-\langle w^2\partial_\beta[Ag], \partial_\beta g\rangle \geqq \left|\partial_\beta g\right|^2_{\sigma,\ell,\vartheta} - \eta \sum_{|\bar\beta|=|\beta|} \left|\partial_{\bar\beta} g\right|^2_{\sigma,\ell,\vartheta} - C(\eta) \sum_{|\bar\beta|<|\beta|} \left|\partial_{\bar\beta} g\right|^2_{\sigma,|\bar\beta|-l,\vartheta},$$

where $w^2 = w^2(\ell,\vartheta)$.

Notice that the estimate involving $[Ag]$ is much weaker than the analogous estimate [9, Lemma 6, p.403] with no exponential weight. In [9], there are no derivatives in the last term on the right. The key problem here is that derivatives of the exponential weight, in particular $\partial_i(w^2(\ell,\vartheta))$, can grow faster than $w^2(\ell,\vartheta)$. Then, in some cases, we do not have enough decay to get the sharper estimate. Instead, we weaken the estimate and use lower order derivatives to extract polynomial decay from higher order weights. For the estimate involving $[Kg_1]$ the difference is the same as in the previous cases; we again show that K controls an exponentially growing factor of $w(\ell,\vartheta)(v)$. We remark that these estimates are not at all optimal. It is not hard to see that you can use a smaller norm over a compact region, in particular, for the terms with no derivatives in the $[Ag]$ estimate.

Proof. We begin with the estimate involving $\partial_\beta[Ag]$. Using Lemma 6, we have

$$\begin{aligned}\langle w^2\partial_\beta[Ag], \partial_\beta g\rangle = &-|\partial_\beta g|^2_{\sigma,\ell,\vartheta} - C_\beta^{\beta_1}\langle w^2\partial_{\beta_1}\sigma^{ij}\partial_{\beta-\beta_1}\partial_j g, \partial_\beta\partial_i g\rangle\\ &-C_\beta^{\beta_2}\langle \partial_i(w^2)\partial_{\beta_2}\sigma^{ij}\partial_{\beta-\beta_2}\partial_j g, \partial_\beta g\rangle\\ &-C_\beta^{\beta_1}\langle w^2\partial_{\beta_1}\{\sigma^{ij}v_iv_j\}\partial_{\beta-\beta_1} g, \partial_\beta g\rangle\\ &+C_\beta^{\beta_2}\langle w^2\partial_{\beta_2}\partial_i\sigma^i\partial_{\beta-\beta_2} g, \partial_\beta g\rangle.\end{aligned} \tag{58}$$

Here summations are over $\beta \geqq \beta_1 > 0$ and $\beta \geqq \beta_2 \geqq 0$. We will estimate each of these terms separately.

Case 1: *The Last Two Terms.*

First, we consider the last two terms in (58). We claim that

$$|w^2\partial_{\beta_2}\partial_i\sigma^i(v)| + |w^2\partial_{\beta_1}\{\sigma^{ij}v_iv_j\}| \leqq C[1+|v|]^{\gamma+1}w^2.$$

For the first term on the left-hand side, this follows from Lemma 4. For the estimate for the second term on the right-hand side, from (9) and (50) we have

$$\sigma^{ij}(v)v_iv_j = \int_{\mathbb{R}^3}\phi^{ij}(v-u)u_iu_j\mu(u)du.$$

Now the estimate follows from [9, Lemma 2, p.397] and $|\beta_1| > 0$. Using the claim, the last two terms in (58) are bounded by

$$C\int w^2[1+|v|]^{\gamma+1}\{|\partial_{\beta-\beta_1}g|+|\partial_{\beta-\beta_2}g|\}|\partial_\beta g| = C\int_{|v|\leqq m} + C\int_{|v|\geqq m}$$
$$\leqq C\int_{|v|\leqq m} + \frac{C}{m}\left|[1+|v|]^{\frac{\gamma+2}{2}}\{|\partial_{\beta-\beta_1}g|+|\partial_{\beta-\beta_2}g|\}\right|_{\ell,\vartheta}\left|[1+|v|]^{\frac{\gamma+2}{2}}\partial_\beta g\right|_{\ell,\vartheta}$$
$$\leqq C\int_{|v|\leqq m} + \frac{C}{m}\sum_{|\bar\beta|\leqq|\beta|}\left|\partial_{\bar\beta}g\right|_{\sigma,\ell,\vartheta}\left|\partial_\beta g\right|_{\sigma,\ell,\vartheta}. \tag{59}$$

We have used Lemma 5 in the last step. For the part $|v| \leqq m$, for any $m' > 0$, by (21), we use the compact Sobolev space interpolation and Lemma 5 to get

$$\int_{|v|\leqq m} \leqq \frac{1}{m'}\sum_{|\bar\beta|=|\beta|+1}\int_{|v|\leqq m}|\partial_{\bar\beta}g|^2 + C_{m'}\int_{|v|\leqq m}|g|^2$$
$$\leqq \frac{C}{m'}\sum_{|\bar\beta|=|\beta|}\left|\partial_{\bar\beta}g\right|^2_{\sigma,\ell} + C_{m'}\,|\bar\chi_m g|^2_\ell. \tag{60}$$

We used Lemma 5 again in the last step. This completes the estimate for the last two terms in (58).

Case 2: *The Third Term.*

Next consider the most delicate third term in (58) when $|\beta_2| = 0$. Recall $\partial_i(w^2)$ from (54); from (55) we have $|w_1(v)| \leqq C$ since $0 \leqq \vartheta \leqq 2$. And from (44) we have

$$\sigma^{ij}(v)v_i = \lambda_1(v)v_j. \tag{61}$$

Using Lemma 4 for the decay of $\lambda_1(v)$, the third term in (58) with $|\beta_2| = 0$ is

$$\left|\langle\partial_i(w^2)\sigma^{ij}\partial_\beta\partial_j g, \partial_\beta g\rangle\right| \leqq C\int w^2(\ell,\vartheta)[1+|v|]^{\gamma+1}\left|\partial_\beta\partial_j g\right|\left|\partial_\beta g\right| dv,$$
$$= C\int\left(w(\ell,\vartheta)[1+|v|]^{\frac{\gamma}{2}}\left|\partial_\beta\partial_j g\right|\right)\left(w(\ell,\vartheta)[1+|v|]^{(\gamma+2)/2}\left|\partial_\beta g\right|\right)dv.$$

Consider the second term in parenthesis. We will use the weight to extract extra polynomial decay and look at this as a term with lower order derivatives in the σ norm. Write $\partial_\beta = \partial_{\beta-e_k}\partial_k$ where e_k is an element of the standard basis. Further, from (10) with $\tau \leqq -1$, write out

$$w(\ell,\vartheta) = (1+|v|^2)^{\tau\ell/2}\exp\left(\frac{q}{4}(1+|v|^2)^{\frac{\vartheta}{2}}\right) = w(\ell-1,\vartheta)(1+|v|^2)^{\tau/2}$$
$$\leqq Cw(\ell-1,\vartheta)[1+|v|]^{-1}.$$

Since $|e_k| = 1$ and $\ell = r - l$ with $r \geqq |\beta|$, $\ell - 1 \geqq |\beta| - 1 - l = |\beta - e_k| - l$. Thus,

$$w(\ell-1,\vartheta)[1+|v|]^{-1} \leqq w(|\beta-e_k|-l,\vartheta)[1+|v|]^{-1}.$$

Hence

$$w(\ell,\vartheta)[1+|v|]^{(\gamma+2)/2} \leqq w(|\beta-e_k|-l,\vartheta)[1+|v|]^{\frac{\gamma}{2}}.$$

Then, for any large $m' > 0$, $\left|\langle \partial_i (w^2)\sigma^{ij}\partial_\beta\partial_j g, \partial_\beta g\rangle\right|$ is

$$
\begin{aligned}
&\leqq C\int \left(w(\ell,\vartheta)[1+|v|]^{\frac{\gamma}{2}}\left|\partial_\beta\partial_j g\right|\right)\\
&\quad\times\left(w(|\beta-e_k|-l,\vartheta)[1+|v|]^{\frac{\gamma}{2}}\left|\partial_{\beta-e_k}\partial_k g\right|\right)dv\\
&\leqq C|\partial_\beta g|_{\sigma,\ell,\vartheta}|\partial_{\beta-e_k} g|_{\sigma,|\beta-e_k|-l,\vartheta}\\
&\leqq \frac{1}{m'}|\partial_\beta g|^2_{\sigma,\ell,\vartheta}+C_{m'}\sum_{|\bar\beta|<|\beta|}|\partial_{\bar\beta} g|^2_{\sigma,|\bar\beta|-l,\vartheta}.
\end{aligned}\tag{62}
$$

This completes the estimate for the third term in (58) when $|\beta_2| = 0$.

Next consider the third term in (58) when $|\beta_2| > 0$. Since $|\beta_2| \geqq 1$,

$$
\left|\partial_i(w^2(\ell,\vartheta))\partial_{\beta_2}\sigma^{ij}\right|\leqq Cw^2(\ell,\vartheta)[1+|v|]^{\gamma+2}.
$$

Notice that the order of $\partial_{\beta-\beta_2}\partial_j$ in this case is $< |\beta|$. Again we exploit the lower order derivative to gain some decay from the weight. Since $\tau \leqq -1$, we split

$$
\begin{aligned}
w(\ell,\vartheta)=w(\ell-1+1,\vartheta)&=w(\ell-1,\vartheta)(1+|v|^2)^{\tau/2}\\
&\leqq w(|\beta-\beta_2|-l,\vartheta)[1+|v|]^{-1}.
\end{aligned}\tag{63}
$$

In this last step we have used $\ell = r - l$, $r \geqq |\beta|$ so that $r - 1 \geqq |\beta - \beta_2|$ since $|\beta_2| \geqq 1$. Given $m' > 0$, in this case, the third term in (58) has the upper bound

$$
\begin{aligned}
&C\int\left|\partial_i(w^2(\ell,\vartheta))\partial_{\beta_2}\sigma^{ij}\right|\left|\partial_{\beta-\beta_2}\partial_j g\partial_\beta g\right|\leqq C\int w^2[1+|v|]^{\gamma+2}\left|\partial_{\beta-\beta_2}\partial_j g\partial_\beta g\right|\\
&\leqq C\left|\partial_\beta g\right|_{\sigma,\ell,\vartheta}\left\{\int w^2(\ell,\vartheta)[1+|v|]^{\gamma+2}\left|\partial_{\beta-\beta_2}\partial_j g\right|^2dv\right\}^{1/2}\\
&\leqq C\left|\partial_\beta g\right|_{\sigma,\ell,\vartheta}\left\{\int w^2(|\beta-\beta_2|-l,\vartheta)[1+|v|]^{\gamma}\left|\partial_{\beta-\beta_2}\partial_j g\right|^2dv\right\}^{1/2}\\
&\leqq C\left|\partial_\beta g\right|_{\sigma,\ell,\vartheta}\sum_{|\bar\beta|\leqq|\beta|-1}\left|\partial_{\bar\beta} g\right|_{\sigma,|\bar\beta|-l,\vartheta}\\
&\leqq\frac{1}{m'}\left|\partial_\beta g\right|^2_{\sigma,\ell,\vartheta}+C_{m'}\sum_{|\bar\beta|\leqq|\beta|-1}\left|\partial_{\bar\beta} g\right|^2_{\sigma,|\bar\beta|-l,\vartheta}.
\end{aligned}\tag{64}
$$

This completes the estimate for the third term in (58). By combining (59), (60), (62), (64) and (65) with m and m' chosen large enough we complete this estimate.

We now estimate $\langle w^2\partial_\beta[Kg_1], g_2\rangle$. Recalling (48), we have

$$
\begin{aligned}
w^2\partial_\beta Kg_1=&-\partial_i\left[w^2\partial_\beta\left\{\mu^{1/2}\left[\phi^{ij}*\left\{\mu^{1/2}\partial_j g_1+\frac{v_j}{2}\mu^{1/2}g_1\right\}\right]\right\}\right]\\
&+\partial_i(w^2)\partial_\beta\left\{\mu^{1/2}\left[\phi^{ij}*\left\{\mu^{1/2}\partial_j g_1+\frac{v_j}{2}\mu^{1/2}g_1\right\}\right]\right\}\\
&+w^2\partial_\beta\left\{\frac{v_i}{2}\mu^{1/2}\left[\phi^{ij}*\left\{\mu^{1/2}\partial_j g_1+\frac{v_j}{2}\mu^{1/2}g_1\right\}\right]\right\}.
\end{aligned}
$$

We take derivatives only on the factor $\{\mu^{1/2}\partial_j g_1 + v_j \mu^{1/2} g_1\}$ in the convolutions above. Upon integrating by parts for the first term, using (54) and (55) and collecting terms we can express $\langle w^2 \partial_\beta [K g_1], g_2 \rangle$ as

$$\sum_{|\beta_1| \leqq |\beta|+1, |\beta_2| \leqq 1} \int w^2 \phi^{ij}(v - v') \mu^{1/2}(v) \mu^{1/2}(v') \bar{\mu}^{\beta_1 \beta_2}(v, v') \times \partial_{\beta_1} g_1(v') \partial_{\beta_2} g_2(v) dv' dv,$$

where $\bar{\mu}^{\beta_1\beta_2}(v, v')$ is a collection of smooth functions which, for any $k-$th order derivatives, satisfies

$$|\nabla^k_{v,v'} \bar{\mu}^{\beta_1\beta_2}(v', v)| \leqq C(1 + |v|^2)^{|\beta|/2}(1 + |v'|^2)^{|\beta|/2}.$$

Using the same argument as in (56) for the same $0 < q' < 1$ as in (56) we can rewrite $\langle w^2 \partial_\beta [K g_1], g_2 \rangle$ as

$$\sum_{|\beta_1| \leqq |\beta|+1, |\beta_2| \leqq 1} \int w(v) \phi^{ij}(v - v') \mu^{q'/4}(v) \mu^{1/4}(v') \mu^{\beta_1\beta_2}(v, v') \times \partial_{\beta_1} g_1(v') \partial_{\beta_2} g_2(v) dv' dv,$$

where $\mu^{\beta_1\beta_2}(v, v')$ is a collection of smooth functions satisfying (for any $k-$th order derivatives)

$$|\nabla^k_{v,v'} \mu^{\beta_1\beta_2}(v', v)| \leqq C e^{-\frac{q'}{16}|v|^2} e^{-\frac{1}{16}|v'|^2}.$$

We split $\langle w^2 \partial_\beta [K g_1], g_2 \rangle$ as in (57) to get

$$\sum \int w(v) \psi^{ij}(v, v') \mu^{\beta_1\beta_2}(v, v') \partial_{\beta_1} g_1(v') \partial_{\beta_2} g_2(v) dv' dv + \sum \int w(v) \{\phi^{ij}(v - v') \mu^{q'/4}(v) \mu^{1/4}(v') - \psi^{ij}\} \times \mu^{\beta_1\beta_2}(v, v') \partial_{\beta_1} g_1(v') \partial_{\beta_2} g_2(v) dv' dv.$$

Using the estimates as for J_2 in (57) and Lemma 5 the last term is bounded by

$$\frac{C}{m} \sum_{|\bar{\beta}| \leqq |\beta|} \left| \partial_{\bar{\beta}} g_1 \right|_{\sigma,\ell} |g_2|_{\sigma,\ell,\vartheta} \cdot$$

Since ψ^{ij} has compact support, integrating by parts over v' and v, the first term is equal to

$$\sum_{|\beta_1| \leqq |\beta|+1, |\beta_2| \leqq 1} (-1)^{|\beta_1|+|\beta_2|} \int \partial_{\beta_2} \{ w(v) \partial_{\beta_1} [\psi^{ij}(v, v') \bar{\mu}_{\beta_1\beta_2}(v, v')] \} \times g_1(v') g_2(v) dv' dv.$$

Then, by Cauchy–Schwartz, this term is $\leqq C(m) \left| \bar{\chi}_{C(m)} \mu g_1 \right|_\ell \left| \bar{\chi}_{C(m)} g_2 \right|_\ell$. This concludes case 2.

Case 3: *The Second Term.*

Next, we consider the second term in (58). Since $|\beta_1| \geqq 1$, we have

$$|\langle w^2 \partial_{\beta_1} \sigma^{ij} \partial_{\beta-\beta_1} \partial_j g, \partial_\beta \partial_i g \rangle| \leqq C \int [1+|v|]^{\gamma+1} w^2 |\partial_{\beta-\beta_1} \partial_j g \partial_\beta \partial_i g|$$
$$\leqq C \left|\partial_\beta g\right|_{\boldsymbol{\sigma},\ell,\vartheta} \left\{ \int [1+|v|]^{\gamma+2} w^2(\ell,\vartheta) |\partial_{\beta-\beta_1} \partial_j g|^2 \right\}^{1/2}.$$

Now using (63) with $\beta_1 = \beta_2$, given $m' > 0$ this is

$$\leqq C \left|\partial_\beta g\right|_{\boldsymbol{\sigma},\ell,\vartheta} \left\{ \int [1+|v|]^{\gamma} w^2(|\beta-\beta_1| - l, \vartheta) |\partial_{\beta-\beta_1} \partial_j g|^2 \right\}^{1/2} \quad (65)$$
$$\leqq C \left|\partial_\beta g\right|_{\boldsymbol{\sigma},\ell,\vartheta} \sum_{|\bar\beta| \leqq |\beta|-1} \left|\partial_{\bar\beta} g\right|_{\boldsymbol{\sigma},|\bar\beta|-l,\vartheta} \leqq \frac{1}{m'} \left|\partial_\beta g\right|^2_{\boldsymbol{\sigma},\ell,\vartheta}$$
$$+ C_{m'} \sum_{|\bar\beta| \leqq |\beta|-1} \left|\partial_{\bar\beta} g\right|^2_{\boldsymbol{\sigma},|\bar\beta|-l,\vartheta}.$$

We have now estimated all the terms in (58). We conclude Case 3 by first choosing m large enough. Our lemma follows by first choosing $m > 0$ large. $\square$

Next, from Lemma 8 we get a general lower bound for L with high derivatives. We also prove a lower bound for L with no derivatives.

Lemma 9. *Let* $0 \leqq \vartheta \leqq 2$, $q > 0$ *and* $l \geqq 0$ *with* $|\beta| > 0$ *and* $\ell = |\beta| - l$. *If* $\vartheta = 2$ *further restrict* $0 < q < 1$. *Then for* $\eta > 0$ *small enough there exists* $C(\eta) > 0$ *such that*

$$\langle w^2 \partial_\beta [Lg], \partial_\beta g \rangle \geqq \left|\partial_\beta g\right|^2_{\boldsymbol{\sigma},\ell,\vartheta} - \eta \sum_{|\beta_1| = |\beta|} \left|\partial_{\beta_1} g\right|^2_{\boldsymbol{\sigma},\ell,\vartheta} - C(\eta) \sum_{|\beta_1| < |\beta|} \left|\partial_{\beta_1} g\right|^2_{\boldsymbol{\sigma},|\beta_1|-l,\vartheta},$$

where $w^2 = w^2(\ell,\vartheta)$. *If* $|\beta| = 0$ *we have*

$$\langle w^2(\ell,\vartheta)[Lg], g \rangle \geqq \delta_q^2 \, |g|^2_{\boldsymbol{\sigma},\ell,\vartheta} - C(\eta) \left|\bar\chi_{C(\eta)} g\right|^2_\ell,$$

where $\delta_q = 1 - q^2 - \eta > 0$ *for* $\eta > 0$ *small enough or* $\delta_q = 1 - \eta > 0$ *if* $\vartheta < 2$.

It turns out that the lower bound for L with no extra v derivatives and an exponential weight needs a new approach. We need to use exact cancellation to make it work in the $\vartheta = 2$ case.

Proof. By Lemma 8, we need only consider the case with $|\beta| = 0$.

First assume $0 \leqq \vartheta < 2$. In this case, after an integration by parts, (47) gives

$$\langle w^2 Lg, g \rangle = |g|^2_{\boldsymbol{\sigma},\ell,\vartheta} + \langle \partial_i (w^2) \sigma^{ij} \partial_j g, g \rangle - \langle w^2 \partial_i \sigma^i g, g \rangle - \langle w^2 K g, g \rangle.$$

By Lemma 7, the last two terms on the right-hand side satisfy the $|\beta| = 0$ estimate. Thus we only consider $\langle \partial_i (w^2) \sigma^{ij} \partial_j g, g \rangle$. By (54) and (61) we can write

$$\partial_i (w^2(v)) \sigma^{ij}(v) = w^2(v) w_1(v) \lambda_1(v) v_j.$$

From (55), $|w_1(v)| \leqq C(1+|v|^2)^{\frac{\vartheta}{2}-1}$, and by Lemma 4, $\left|\lambda_1(v)v_j\right| \leqq C[1+|v|]^{\gamma+1}$. Thus for any $m'>0$

$$\begin{aligned}\left|\langle\partial_i(w^2)\sigma^{ij}\partial_j g, g\rangle\right| &\leqq C\int w^2(\ell,\vartheta)[1+|v|]^{\gamma+1+\frac{\vartheta}{2}-1}\left|\partial_j g\right||g|\,dv\\ &= C\int\left(w(\ell,\vartheta)[1+|v|]^{\frac{\gamma}{2}}\left|\partial_j g\right|\right)\\ &\quad\times\left(w(\ell,\vartheta)[1+|v|]^{\frac{\gamma+2}{2}+\frac{\vartheta}{2}-1}|g|\right)\\ &\leqq C\,|g|_{\boldsymbol{\sigma},\ell,\vartheta}\left|[1+|v|]^{\frac{\gamma+2}{2}+\frac{\vartheta}{2}-1}g\right|_{\ell,\vartheta}\\ &\leqq \frac{1}{m'}\,|g|^2_{\boldsymbol{\sigma},\ell,\vartheta}+C(m')\left|[1+|v|]^{\frac{\gamma+2}{2}+\frac{\vartheta}{2}-1}g\right|^2_{\ell,\vartheta}.\end{aligned}$$

For another $m>0$ further split

$$\begin{aligned}\left|[1+|v|]^{(\gamma+2)/2+\frac{\vartheta}{2}-1}g\right|^2_{\ell,\vartheta} &= \int_{|v|\leqq m}+\int_{|v|>m}\\ &\leqq \int_{|v|\leqq m}+Cm^{\vartheta-2}\int_{|v|>m}w^2[1+|v|]^{\gamma+2}|g|^2dv\\ &\leqq C(m)\int_{|v|\leqq m}w^2(\ell,0)|g|^2dv+Cm^{\vartheta-2}\,|g|^2_{\boldsymbol{\sigma},\ell,\vartheta}\,.\\ &\leqq C(m)|\bar{\chi}_m g|_\ell+Cm^{\vartheta-2}\,|g|^2_{\boldsymbol{\sigma},\ell,\vartheta}\,.\end{aligned}\tag{66}$$

We thus complete the estimate for $0\leqq\vartheta<2$ by choosing m and m' large.

Finally consider the case $\vartheta=2$ and $0<q<1$. We will prove this case in two steps. Split $L=-A-K$. Define $M(v)\equiv\exp\left(\frac{q}{4}\left(1+|v|^2\right)\right)$. First we will show that there is $\delta_q>0$ such that

$$-\langle w^2(\ell,2)[Ag],g\rangle\geqq\delta_q\,|Mg|^2_{\boldsymbol{\sigma},\ell}-C(\delta_q)\left|\bar{\chi}_{C(\delta_q)}g\right|^2_\ell\,.\tag{67}$$

Second we will establish

$$|Mg|^2_{\boldsymbol{\sigma},\ell}\geqq\delta_q|g|^2_{\boldsymbol{\sigma},\ell,2}-C(\delta_q)|g\bar{\chi}_{C(\delta_q)}|^2_\ell,\tag{68}$$

where $\delta_q=1-q^2-\frac{\eta}{2}>0$ since $\eta>0$ can be chosen arbitrarily small. This will be enough to establish the case $\vartheta=2$ because the K part is controlled by Lemma 7.

We now establish (67). By (47), we obtain

$$
\begin{aligned}
-M[Ag] &= -M\partial_i\{\sigma^{ij}\partial_j g\} + M\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g - M\partial_i\sigma^i g \\
&= -\partial_i\{M\sigma^{ij}\partial_j g\} + q\sigma^{ij}\frac{v_i}{2}M\partial_j g + M\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g - M\partial_i\sigma^i g \\
&= -\partial_i\{\sigma^{ij}\partial_j[Mg]\} + q\partial_i(\sigma^{ij}\frac{v_j}{2}Mg) + q\sigma^{ij}\frac{v_i}{2}M\partial_j g \\
&\quad + M\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g - M\partial_i\sigma^i g \\
&= -\partial_i\{\sigma^{ij}\partial_j[Mg]\} + q\partial_i\{\sigma^i Mg\} + q\sigma^j\partial_j[Mg] \\
&\quad - q^2 M\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g + M\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g - M\partial_i\sigma^i g.
\end{aligned}
$$

Notice that after an integration by parts

$$
\langle(1+|v|^2)^{\tau\ell}\{q\partial_i\{\sigma^i Mg\} + q\sigma^j\partial_j[Mg]\}, Mg\rangle = -q\langle\partial_i(1+|v|^2)^{\tau\ell}\sigma^i Mg, Mg\rangle.
$$

Also, by (10),

$$
-\langle w^2(\ell,2)[Ag], g\rangle = -\langle(1+|v|^2)^{\tau\ell}M[Ag], Mg\rangle.
$$

We therefore have

$$
\begin{aligned}
-\langle w^2(\ell,2)[Ag], g\rangle &= \int (1+|v|^2)^{\tau\ell}\Big\{\sigma^{ij}\partial_j[Mg]\partial_i[Mg] \\
&\qquad +(1-q^2)\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}[Mg]^2\Big\}dv - \int w^2(\ell,2)\partial_i\sigma^i g^2 dv \\
&\qquad + \int \partial_i(1+|v|^2)^{\tau\ell}\Big\{\sigma^{ij}\{\partial_j[Mg]\}[Mg] - q\sigma^i[Mg]^2\Big\}dv \\
&\geqq (1-q^2)|Mg|^2_{\sigma,\ell} - \int w^2(\ell,2)\partial_i\sigma^i g^2 dv \\
&\qquad + \int \partial_i(1+|v|^2)^{\tau\ell}\Big\{\sigma^{ij}\{\partial_j[Mg]\}[Mg] - q\sigma^i[Mg]^2\Big\}dv.
\end{aligned}
$$

By Lemma 4, $\left|\partial_i\sigma^i\right| \leqq C[1+|v|]^{\gamma+1}$. Then as in (66), for $m>0$, we have

$$
\begin{aligned}
\int w^2(\ell,2)\left|\partial_i\sigma^i\right| g^2 dv &= \int_{|v|\leqq m} + \int_{|v|>m} \\
&\leqq \int_{|v|\leqq m} + \frac{C}{m}\int_{|v|>m}(1+|v|^2)^{\tau\ell}[1+|v|]^{\gamma+2}[Mg]^2 dv \\
&\leqq \int_{|v|\leqq m} + \frac{C}{m}|Mg|^2_{\boldsymbol{\sigma},\ell} \\
&\leqq C(m)\,|\bar{\chi}_m g|^2_\ell + \frac{\eta}{4}|Mg|^2_{\boldsymbol{\sigma},\ell}\,,
\end{aligned}
\tag{69}
$$

where the last line follows from choosing $m > 0$ large enough. We integrate by parts on the next term to obtain

$$\int \partial_i (1+|v|^2)^{\tau\ell} \sigma^{ij} \{\partial_j [Mg]\}[Mg] dv = -\frac{1}{2} \int \partial_j \left\{ \partial_i (1+|v|^2)^{\tau\ell} \sigma^{ij} \right\} [Mg]^2 dv.$$

By Lemma 4 and (10),

$$\left| \partial_i (1+|v|^2)^{\tau\ell} \sigma^i \right| + \left| \partial_j \left\{ \partial_i (1+|v|^2)^{\tau\ell} \sigma^{ij} \right\} \right| \leqq C (1+|v|^2)^{\tau\ell} [1+|v|]^{\gamma+1}.$$

Thus the estimate for the final term follows from the same argument as (69). This establishes (67).

We finally establish (68). Notice that

$$|Mg|^2_{\sigma,\ell} = \int (1+|v|^2)^{\tau\ell} \left\{ \sigma^{ij} \partial_i [Mg] \partial_j [Mg] + \sigma^{ij} \frac{v_i}{2} \frac{v_j}{2} [gM]^2 \right\} dv.$$

We expand the first term in $|Mg|^2_{\sigma,\ell}$ to obtain

$$\begin{aligned}
&\int (1+|v|^2)^{\tau\ell} \sigma^{ij} \partial_i [Mg] \partial_j [Mg] dv \\
&= \int (1+|v|^2)^{\tau\ell} M^2 \sigma^{ij} \partial_i g \partial_j g dv \\
&\quad + q^2 \int (1+|v|^2)^{\tau\ell} M^2 \sigma^{ij} \frac{v_i}{2} \frac{v_j}{2} g^2 dv \\
&\quad + 2q \int (1+|v|^2)^{\tau\ell} M^2 \sigma^{ij} \frac{v_i}{2} \{\partial_j g\} g dv \\
&= \int w^2(\ell, 2) \left\{ \sigma^{ij} \partial_i g \partial_j g + q^2 \sigma^{ij} \frac{v_i}{2} \frac{v_j}{2} g^2 \right\} dv \\
&\quad + 2q \int (1+|v|^2)^{\tau\ell} M^2 \sigma^j \{\partial_j g\} g dv.
\end{aligned} \tag{70}$$

In the last step we used (10). We integrate by parts on the last term to obtain

$$\begin{aligned}
2q \int (1+|v|^2)^{\tau\ell} M^2 \sigma^j \{\partial_j g\} g dv &= q \int (1+|v|^2)^{\tau\ell} M^2 \sigma^j \{\partial_j g^2\} dv \\
&= -q \int (1+|v|^2)^{\tau\ell} M^2 \partial_j \sigma^j g^2 dv \\
&\quad - q \int \partial_j (1+|v|^2)^{\tau\ell} M^2 \sigma^j g^2 dv \\
&\quad - 2q^2 \int (1+|v|^2)^{\tau\ell} M^2 \sigma^{ij} \frac{v_i}{2} \frac{v_j}{2} g^2 dv.
\end{aligned}$$

Since $\partial_j (1+|v|^2)^{\tau\ell} = (1+|v|^2)^{\tau\ell} \left\{ 2\tau\ell (1+|v|^2)^{-1} v_j \right\}$, we define the error as

$$\bar{w}(v) = q \left\{ \partial_j \sigma^j + 2\tau\ell (1+|v|^2)^{-1} \sigma^j v_j \right\}.$$

Then (70) is

$$= \int w^2(\ell,2)\left\{\sigma^{ij}\partial_i g\partial_j g - q^2\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g^2\right\}dv - \int w^2(\ell,2)\bar{w}g^2 dv.$$

Adding the second term in $|Mg|^2_{\sigma,\ell}$ to both sides of the last display yields

$$|Mg|^2_{\sigma,\ell} \geqq (1-q^2)|g|^2_{\sigma,\ell,2} - \int w^2(\ell,2)\bar{w}g^2 dv.$$

By Lemma 4 we have

$$|\bar{w}(v)| \leqq C[1+|v|]^{\gamma+1}.$$

Thus, using (69), for any small $\eta > 0$ we have

$$\int w^2(\ell,2)|\bar{w}|g^2 dv \leqq C(m)|g\bar{\chi}_m|^2_\ell + \frac{\eta}{2}|g|^2_{\sigma,\ell,2}.$$

This completes the estimate (68) and the proof.

We thus conclude our estimates for the linear terms and finish the section by estimating the nonlinear term.

Lemma 10. *Let* $|\alpha| + |\beta| \leqq N$, $0 \leqq \vartheta \leqq 2$, $q > 0$ *and* $l \geqq 0$ *with* $\ell = |\beta| - l$. *If* $\vartheta = 2$ *restrict* $0 < q < 1$. *Then*

$$\begin{aligned} &\langle w^2(\ell,\vartheta)\partial^\alpha_\beta\Gamma[g_1,g_2],\ \partial^\alpha_\beta g_3\rangle \qquad (71)\\ &\leqq C\sum\left\{|\partial^{\alpha_1}_{\bar{\beta}}g_1|_\ell|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\sigma,\ell,\vartheta} + |\partial^{\alpha_1}_{\bar{\beta}}g_1|_{\sigma,\ell}|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\ell,\vartheta}\right\}|\partial^\alpha_\beta g_3|_{\sigma,\ell,\vartheta},\end{aligned}$$

where summation is over $|\alpha_1| + |\beta_1| \leqq N$, $\bar{\beta} \leqq \beta_1 \leqq \beta$.

Furthermore,

$$\begin{aligned} &\left(w^2(\ell,\vartheta)\partial^\alpha_\beta\Gamma[g_1,g_2],\ \partial^\alpha_\beta g_3\right) \qquad (72)\\ &\leqq C\left\{\mathcal{E}^{1/2}_l(g_1)\mathcal{D}^{1/2}_{l,\vartheta}(g_2) + \mathcal{D}^{1/2}_l(g_1)\mathcal{E}^{1/2}_{l,\vartheta}(g_2)\right\}\|\partial^\alpha_\beta g_3\|_{\sigma,\ell,\vartheta}.\end{aligned}$$

The proof of Lemma 10 is more or less the same as in [9] except for a few details. The differences mainly come from taking derivatives of the exponential weight $w(\ell,\vartheta)(v)$ which creates extra polynomial growth.

Proof. Recall $\Gamma[g_1,g_2]$ in (49). By the product rule, we expand

$$\langle w^2\partial^\alpha_\beta\Gamma[g_1,g_2],\partial^\alpha_\beta g_2\rangle = \sum C^{\alpha_1}_\alpha C^{\beta_1}_\beta \times G_{\alpha_1\beta_1},$$

where $G_{\alpha_1\beta_1}$ takes the form

$$-\langle w^2\{\phi^{ij} * \partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1} g_1]\}\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2, \partial_i \partial^{\alpha}_{\beta} g_3\rangle \tag{73}$$

$$-\langle w^2 \left\{\phi^{ij} * \partial_{\beta_1}\left[\frac{v_i}{2}\mu^{1/2}\partial^{\alpha_1} g_1\right]\right\} \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2, \partial^{\alpha}_{\beta} g_3\rangle \tag{74}$$

$$+\langle w^2 \left\{\phi^{ij} * \partial_{\beta_1}\left[\mu^{1/2}\partial_j\partial^{\alpha_1} g_1\right]\right\} \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2, \partial_i\partial^{\alpha}_{\beta} g_3\rangle \tag{75}$$

$$+\langle w^2 \left\{\phi^{ij} * \partial_{\beta_1}\left[\frac{v_i}{2}\mu^{1/2}\partial_j\partial^{\alpha_1} g_1\right]\right\} \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2, \partial^{\alpha}_{\beta} g_3\rangle \tag{76}$$

$$-\langle \partial_i[w^2]\{\phi^{ij} * \partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1} g_1]\}\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2, \partial^{\alpha}_{\beta} g_3\rangle \tag{77}$$

$$+\langle \partial_i[w^2]\{\phi^{ij} * \partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1} g_1]\}\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2, \partial^{\alpha}_{\beta} g_3\rangle. \tag{78}$$

The last two terms appear when we integrate by parts over the v_i variable.

We estimate the last term (78) first. Recall from (54) and (55) that $\partial_i[w^2] = w^2(v)w_1(v)v_i$, where $|w_1(v)| \leqq C$. By first summing over i and using (50) we can rewrite (78) as

$$\langle [w^2 w_1]\{\phi^{ij} * v_i\partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1} g_1]\}\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2, \partial^{\alpha}_{\beta} g_3\rangle.$$

Since $-1 \leqq \gamma + 2 < 0$, $\phi^{ij}(v) \in L^2_{loc}(\mathbb{R}^3)$ and $|\partial_{\beta_1}\{\mu^{1/2}\}| \leqq C\mu^{1/4}$, we deduce by the Cauchy–Schwartz inequality and Lemma 5 that

$$\begin{aligned}\{\phi^{ij} * v_i\partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1} g_1]\} &\leqq [|\phi^{ij}|^2 * \mu^{1/8}]^{1/2}(v) \sum_{\bar\beta \leqq \beta_1} |\mu^{1/32}\partial_j\partial^{\alpha_1}_{\bar\beta} g_1|_{\ell}. \\ &\leqq C[1+|v|]^{\gamma+2} \sum_{\bar\beta \leqq \beta_1} \left|\partial^{\alpha_1}_{\bar\beta} g_1\right|_{\sigma,\ell}, \end{aligned} \tag{79}$$

where we have used Lemma 2 in [9] to argue that

$$[|\phi^{ij}|^2 * \mu^{1/8}]^{1/2}(v) \leqq C[1+|v|]^{\gamma+2}.$$

Using the above, (78) is bounded by Lemma 5 as

$$\begin{aligned} &C\sum_{\bar\beta \leqq \beta_1} \left|\partial^{\alpha_1}_{\bar\beta} g_1\right|_{\sigma,\ell} \int w^2(\ell,\vartheta)[1+|v|]^{\gamma+2}|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 \partial^{\alpha}_{\beta} g_3| dv \\ &\leqq C\sum_{\bar\beta \leqq \beta_1} |\partial^{\alpha_1}_{\bar\beta} g_1|_{\sigma,\ell} |\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2|_{\ell,\vartheta} |[1+|v|]^{\gamma+2}\partial^{\alpha}_{\beta} g_3|_{\ell,\vartheta} \\ &\leqq C\sum_{\bar\beta \leqq \beta_1} |\partial^{\alpha_1}_{\bar\beta} g_1|_{\sigma,\ell} |\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2|_{\ell,\vartheta} |\partial^{\alpha}_{\beta} g_3|_{\sigma,\ell,\vartheta}. \end{aligned}$$

This completes the estimate for (78).

We now estimate (73)-(77). We decompose their double integration region $[v, v'] \in \mathbb{R}^3 \times \mathbb{R}^3$ into three parts

$$\{|v| \leqq 1\}, \quad \{2|v'| \geqq |v|,\ |v| \geqq 1\} \quad \text{and} \quad \{2|v'| \leqq |v|,\ |v| \geqq 1\}.$$

Case 1: *Terms* (73)-(77) *over* $\{|v| \leqq 1\}$.

For the first part $\{|v| \leqq 1\}$, recall $\phi^{ij}(v) = O(|v|^{\gamma+2}) \in L^2_{loc}$. As in (79), we have

$$|\phi^{ij} * \partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]| + |\phi^{ij} * \partial_{\beta_1}[v_i\mu^{1/2}\partial^{\alpha_1}g_1]| + |\phi^{ij} * v_i\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]|$$
$$\leqq C[1+|v|]^{\gamma+2}\sum_{\bar\beta\leqq\beta}|\partial^{\alpha_1}_{\bar\beta}g_1|_\ell,$$
$$|\phi^{ij} * \partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]| + |\phi^{ij} * \partial_{\beta_1}[\{v_i\mu^{1/2}\}\partial_j\partial^{\alpha_1}g_1]|$$
$$\leqq C[1+|v|]^{\gamma+2}\sum_{\bar\beta\leqq\beta}|\partial^{\alpha_1}_{\bar\beta}g_1|_{\sigma,\ell}.$$

Hence their corresponding integrands over the region $\{|v| \leqq 1\}$ are bounded by

$$C\sum\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_\ell[1+|v|]^{\gamma+2}|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|\left[|\partial_i\partial^\alpha_\beta g_3| + |\partial^\alpha_\beta g_3|\right]$$
$$+C\sum\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_{\sigma,\ell}[1+|v|]^{\gamma+2}|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|\left[|\partial_i\partial^\alpha_\beta g_3| + |\partial^\alpha_\beta g_3|\right],$$

whose $v-$integral over $\{|v| \leqq 1\}$ is clearly bounded by the right-hand side of (71). We thus conclude the first part of $\{|v| \leqq 1\}$ for (73)–(77).

Case 2: *Terms* (73)-(77) *over* $\{2|v'| \geqq |v|, |v| \geqq 1\}$.

For the second part $\{2|v'| \geqq |v|, |v| \geqq 1\}$, we have

$$|\partial_{\beta_1}\mu^{1/2}(v')| + |\partial_{\beta_1}\{v'_j\mu^{1/2}(v')\}| \leqq C\mu^{1/8}(v')\mu^{1/32}(v).$$

Thus, by the same type of estimates as in (79), using the region, we have

$$|\phi^{ij} * \partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]| + |\phi^{ij} * \partial_{\beta_1}[v_i\mu^{1/2}\partial^{\alpha_1}g_1]| + |\phi^{ij} * v_i\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]|$$
$$\leqq C[1+|v|]^{\gamma+2}\mu^{1/64}(v)\sum_{\bar\beta\leqq\beta}|\partial^{\alpha_1}_{\bar\beta}g_1|_\ell,$$
$$|\phi^{ij} * \partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]| + |\phi^{ij} * \partial_{\beta_1}[\{v_i\mu^{1/2}\}\partial_j\partial^{\alpha_1}g_1]|$$
$$\leqq C[1+|v|]^{\gamma+2}\mu^{1/64}(v)\sum_{\bar\beta\leqq\beta}|\partial^{\alpha_1}_{\bar\beta}g_1|_{\sigma,\ell}.$$

And then the $v-$ integrands in (73) to (77) over this region are bounded by

$$C\sum\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_\ell w^2(\ell,\vartheta)[1+|v|]^{\gamma+2}\mu^{1/64}(v)|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|\left[|\partial_i\partial^\alpha_\beta g_3| + |\partial^\alpha_\beta g_3|\right]$$
$$+C\sum\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_{\sigma,\ell}w^2(\ell,\vartheta)[1+|v|]^{\gamma+2}\mu^{1/64}(v)|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|\left[|\partial_i\partial^\alpha_\beta g_3| + |\partial^\alpha_\beta g_3|\right].$$

By Lemma 5, its $v-$integral is bounded by the right-hand side of (71) because of the fast decaying factor $\mu^{1/64}(v)$. We thus conclude the estimate for the second region $\{2|v'| \geqq |v|, |v| \geqq 1\}$ for (73) to (77).

Case 3: *Terms* (73)-(77) *over* $\{2|v'| \leqq |v|, |v| \geqq 1\}$.

We finally consider the third part of $\{2|v'| \leqq |v|, |v| \geqq 1\}$, for which we shall estimate each term from (73) to (77). The key is to Taylor expand $\phi^{ij}(v-v')$. To estimate (73) over the this region we expand $\phi^{ij}(v-v')$ to get

$$\phi^{ij}(v-v') = \phi^{ij}(v) - \sum_k \partial_k \phi^{ij}(v) v'_k + \frac{1}{2} \sum_{k,l} \partial_{kl} \phi^{ij}(\bar{v}) v'_k v'_l, \tag{80}$$

where $\bar{v}$ is between v and $v - v'$. We plug (80) into the integrand of (73). From (42), (44) and (50), we can decompose $\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2$ and $\partial_i \partial^{\alpha}_{\beta} g_3$ into their P_v parts as well as $I - P_v$ parts. For the first term in the expansion (80) we have

$$\sum_{ij} \phi^{ij}(v) \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2(v) \partial_i \partial^{\alpha}_{\beta} g_3(v)$$
$$= \sum_{ij} \phi^{ij}(v) \{[I - P_v] \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2(v)\}\{[I - P_v] \partial_i \partial^{\alpha}_{\beta} g_3(v)\}.$$

Here we have used (50) so that sum of terms with either $P_v \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2$ or $P_v \partial_i \partial^{\alpha}_{\beta} g_3$ vanishes. The absolute value of this is bounded by

$$C[1 + |v|]^{\gamma+2} |[I - P_v] \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2(v)| \times |[I - P_v] \partial_i \partial^{\alpha}_{\beta} g_3(v)|. \tag{81}$$

For the second term in the expansion (80), by taking a k derivative of

$$\sum_{i,j} \phi^{ij}(v) v_i v_j = 0$$

we have

$$\sum_{i,j} \partial_k \phi^{ij}(v) v_i v_j = -2 \sum_j \phi^{kj}(v) v_j = 0.$$

Therefore

$$\sum_{i,j} \partial_k \phi^{ij}(v) P_v \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 P_v \partial_i \partial^{\alpha}_{\beta} g_3 = 0.$$

Splitting $\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2$ and $\partial_i \partial^{\alpha}_{\beta} g_3$ into their P_v and $I - P_v$ parts yields

$$\sum_{i,j} \partial_k \phi^{ij}(v) \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2(v) \partial_i \partial^{\alpha}_{\beta} g_3(v)$$
$$= \sum_{i,j} \partial_k \phi^{ij}(v) \{[P_v \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2][I - P_v] \partial_i \partial^{\alpha}_{\beta} g_3$$
$$+ [I - P_v] \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 [P_v \partial_i \partial^{\alpha}_{\beta} g_3]\} + \sum_{i,j} \partial_k \phi^{ij}(v) [I - P_v] \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 [I - P_v] \partial_i \partial^{\alpha}_{\beta} g_3.$$

Notice that $|\partial_k \phi^{ij}(v)| \leqq C[1 + |v|]^{\gamma+1}$ for $|v| \geqq 1$, we therefore majorize the above by

$$C[1 + |v|]^{\gamma/2} \{|P_v \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2| + |P_v \partial_i \partial^{\alpha}_{\beta} g_3|\}$$
$$\times [1 + |v|]^{(\gamma+2)/2} \{|[I - P_v] \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2| + |[I - P_v] \partial_i \partial^{\alpha}_{\beta} g_3|\} \tag{82}$$
$$+ C[1 + |v|]^{\gamma+1} |[I - P_v] \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2| |[I - P_v] \partial_i \partial^{\alpha}_{\beta} g_3|.$$

Next, we estimate the third term in (80). Using the region we have

$$\frac{1}{2}|v| \leqq |v| - |v'| \leqq |\bar{v}| \leqq |v'| + |v| \leqq \frac{3}{2}|v|. \tag{83}$$

Thus

$$|\partial_{kl}\phi^{ij}(\bar{v})| \leqq C[1+|v|]^{\gamma},$$

and

$$\left|\partial_{kl}\phi^{ij}(\bar{v})\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(v)\partial_i\partial_\beta^\alpha g_3(v)\right| \leqq C[1+|v|]^{\gamma}|\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\partial_i\partial_\beta^\alpha g_3|. \tag{84}$$

Combining (80), (81), (82) and (84) we obtain the estimate

$$\begin{aligned}
&\left|\sum_{i,j}\phi^{ij}(v-v')\partial_i\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\partial_i\partial_\beta^\alpha g_3\right| \\
&\leqq C[1+|v'|]^2\left|\sum_{i,j}\phi^{ij}(v)\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(v)\partial_i\partial_\beta^\alpha g_3(v)\right| \\
&+C[1+|v'|]^2\left|\sum_{i,j}\partial_k\phi^{ij}(v)\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(v)\partial_i\partial_\beta^\alpha g_3(v)\right| \\
&+C[1+|v'|]^2\sum_{i,j}\left|\partial_{kl}\phi^{ij}(\bar{v})\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(v)\partial_i\partial_\beta^\alpha g_3(v)\right| \\
&\leqq C[1+|v'|]^2\{\sigma^{ij}\partial_i\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\}^{1/2}\{\sigma^{ij}\partial_i\partial_\beta^\alpha g_3\partial_j\partial_\beta^\alpha g_3\}^{1/2},
\end{aligned}$$

where we have used (45) in the last line. The v integrand over $\{2|v'| \leqq |v|, |v| \geqq 1\}$ in (73) is thus bounded by

$$\begin{aligned}
&w^2\int[1+|v'|]^2\mu^{1/4}(v')|\partial_{\bar{\beta}}^{\alpha_1}g_1(v')|dv' \\
&\times\{\sigma^{ij}\partial_i\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\}^{1/2}\{\sigma^{ij}\partial_i\partial_\beta^\alpha g_3\partial_j\partial_\beta^\alpha g_3\}^{1/2} \\
&\leqq C|\partial_{\bar{\beta}}^\alpha g_1|_\ell\{w^2\sigma^{ij}\partial_i\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\}^{1/2}\{w^2\sigma^{ij}\partial_i\partial_\beta^\alpha g_3\partial_j\partial_\beta^\alpha g_3\}^{1/2}.
\end{aligned}$$

Its further integration over v is bounded by the right-hand side of (71).

We now consider the second term (74). We again expand $\phi^{ij}(v-v')$ as

$$\phi^{ij}(v-v') = \phi^{ij}(v) - \sum_k\partial_k\phi^{ij}(\bar{v})v_k', \tag{85}$$

with $\bar{v}$ between v and $v-v'$. Since $\sum_j\phi^{ij}(v)v_j = 0$ we obtain as before

$$\begin{aligned}
&\sum_j\phi^{ij}(v)\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(v)\partial_\beta^\alpha g_3(v) = \sum_j\phi^{ij}(v)\{I-P_v\}\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(v)\times\partial_\beta^\alpha g_3(v) \\
&\leqq C[1+|v|]^{\gamma+2}|\{I-P_v\}\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(v)||\partial_\beta^\alpha g_3(v)| \qquad (86) \\
&\leqq C|[1+|v|]^{\frac{\gamma+2}{2}}\{I-P_v\}\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(v)||[1+|v|]^{\frac{\gamma+2}{2}}\partial_\beta^\alpha g_3(v)| \\
&\leqq C\{\sigma^{ij}\partial_i\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\}^{1/2}\{\sigma^{ij}v_iv_j|\partial_\beta^\alpha g_3|^2\}^{1/2},
\end{aligned}$$

where we have used (45). From (83), $|\partial_k\phi^{ij}(\bar{v})| \leqq C[1+|v|]^{\gamma+1}$. Hence

$$
\begin{aligned}
&|\partial_k\phi^{ij}(\bar{v})\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial^{\alpha}_{\beta}g_3(v)| \qquad (87)\\
&\quad\leqq C[1+|v|]^{\gamma+1}\{|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)|\}\{|\partial^{\alpha}_{\beta}g_3(v)|\}\\
&\quad\leqq C\{[1+|v|]^{\gamma/2}|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)|\}\{[1+|v|]^{\frac{\gamma+2}{2}}|\partial^{\alpha}_{\beta}g_3(v)|\}\\
&\quad\leqq C\{\sigma^{ij}\partial_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\}^{1/2}\{\sigma^{ij}v_iv_j|\partial^{\alpha}_{\beta}g_3|^2\}^{1/2}.
\end{aligned}
$$

From (86) and (87), we thus conclude

$$
\left|\sum_{ij}\phi^{ij}(v-v')\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial^{\alpha}_{\beta}g_3(v)\right|
$$
$$
\leqq C[1+|v'|]\{\sigma^{ij}\partial_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\}^{1/2}\{\sigma^{ij}v_iv_j|\partial^{\alpha}_{\beta}g_3|^2\}^{1/2}.
$$

We thus conclude that the v integrand in (74) can be majorized by

$$
\begin{aligned}
&C\sum\int[1+|v'|]\mu^{1/4}(v')|\partial^{\alpha_1}_{\bar{\beta}}g_1(v')|dv'\\
&\times w^2\{\sigma^{ij}\partial_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\}^{1/2}\{\sigma^{ij}\partial_i\partial^{\alpha}_{\beta}g_3\partial_j\partial^{\alpha}_{\beta}g_3\}^{1/2}\\
&\leqq C\sum|\partial^{\alpha_1}_{\bar{\beta}}g_1|_{\ell}\{w^2\sigma^{ij}\partial_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\}^{1/2}\{w^2\sigma^{ij}\partial_i\partial^{\alpha}_{\beta}g_3\partial_j\partial^{\alpha}_{\beta}g_3\}^{1/2}.
\end{aligned}
$$

Further integration over v shows that this is bounded by the right-hand side of (71).

We now consider the third term (75) over $\{2|v'|\leqq|v|, |v|\geqq 1\}$. We use an integration by parts inside the convolution to split (75) into two parts

$$
\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1] = \partial_j\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]-\phi^{ij}*\partial_{\beta_1}[\partial_j\mu^{1/2}\ \partial^{\alpha_1}g_1]. \qquad (88)
$$

Recall expansion (85) and decompose

$$
\partial_i\partial^{\alpha}_{\beta}g_3 = P_v\partial_i\partial^{\alpha}_{\beta}g_3 + [I-P_v]\partial_i\partial^{\alpha}_{\beta}g_3.
$$

By similar estimates to (86) and (87), the second part of (75) can be estimated as

$$
\begin{aligned}
&\int_{\{|v|\geqq 1,2|v'|\leqq|v|\}}|w^2\,\phi^{ij}(v-v')\partial_{\beta_1}[\partial_j\mu^{1/2}(v')\partial^{\alpha_1}g_1(v')]\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial_i\partial^{\alpha}_{\beta}g_3(v)|\\
&=\int_{|v|\geqq 1,2|v'|\leqq v|}|w^2[\phi^{ij}(v)-\partial_k\phi^{ij}(\bar{v})v'_k]\partial_{\beta_1}[\partial_j\mu^{1/2}(v')\partial^{\alpha_1}g_1(v')]\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_i\partial^{\alpha}_{\beta}g_3|\\
&\leqq C\left|\partial^{\alpha_1}_{\bar{\beta}}g_1\right|_{\ell}\left|[1+|v|]^{\frac{\gamma+2}{2}}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\right|_{\ell,\vartheta}\left|w^{\vartheta}[1+|v|]^{\frac{\gamma+2}{2}}\{I-P_v\}\partial_i\partial^{\alpha}_{\beta}g_3\right|_{\ell,\vartheta}\\
&\quad+C\left|\partial^{\alpha_1}_{\bar{\beta}}g_1\right|_{\ell}\left|[1+|v|]^{\frac{\gamma+2}{2}}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\right|_{\ell,\vartheta}\left|[1+|v|]^{\gamma/2}\partial_i\partial^{\alpha}_{\beta}g_3\right|_{\ell,\vartheta}.
\end{aligned}
$$

By Lemma 5, this is bounded by the right-hand side of (71).

For the first part of (75) by (88) notice that our integration region implies

$$
|\partial_j\phi^{ij}(v-v')|\leqq C[1+|v|]^{\gamma+1}.
$$

We thus have

$$\int_{\{|v|\geqq 1, 2|v'|\leqq |v|\}} w^2 \, |\partial_j \phi^{ij}(v-v')| \partial_{\beta_1}[\mu^{1/2}(v')\partial^{\alpha_1} g_1(v')] \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2(v) \partial_i \partial^{\alpha}_{\beta} g_3(v)$$
$$\leqq C \sum_{\bar\beta \leqq \beta_1} \left|\partial^{\alpha_1}_{\bar\beta} g_1\right|_{\ell} \left|[1+|v|]^{\frac{\gamma+2}{2}} \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2\right|_{\ell,\vartheta} \left|[1+|v|]^{\gamma/2} \partial_j \partial^{\alpha}_{\beta} g_3\right|_{\ell,\vartheta},$$

which is bounded by the right-hand side of (71) by Lemma 5.

Next consider (76) over $\{2|v'| \leqq |v|, |v| \geqq 1\}$. We split (76) as in (88)

$$\phi^{ij} * \partial_{\beta_1}[v_i \mu^{1/2} \partial_j \partial^{\alpha_1} g_1] = \partial_j \phi^{ij} * \partial_{\beta_1}[v_i \mu^{1/2} \partial^{\alpha_1} g_1] - \phi^{ij} * \partial_{\beta_1}[\partial_j \{v_i \mu^{1/2}\} \, \partial^{\alpha_1} g_1].$$

Since $|\phi^{ij}(v-v')| \leqq C[1+|v|]^{\gamma+2}$, and $|\partial_j \phi^{ij}(v-v')| \leqq C[1+|v|]^{\gamma+1}$ (76) is bounded by

$$\int w^2 [1+|v|]^{\gamma+1} |\partial_{\beta_1}[v_i' \mu^{1/2}(v') \partial^{\alpha_1} g_1(v')] \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 \partial^{\alpha}_{\beta} g_3| dv' dv$$
$$+ \int w^2 [1+|v|]^{\gamma+2} |\partial_{\beta_1}[\partial_j \{v_i' \mu^{1/2}(v')\} \partial^{\alpha_1} g_1(v')] \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 \partial^{\alpha}_{\beta} g_3| dv' dv$$
$$\leqq C \sum_{\bar\beta \leqq \beta_1} \left|\partial^{\alpha_1}_{\bar\beta} g_1\right|_{\ell} \left|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2\right|_{\sigma,\ell,\vartheta} \left|\partial^{\alpha}_{\beta} g_3\right|_{\sigma,\ell,\vartheta}.$$

We thus conclude the estimate for (76).

Finally, consider the term (77) over $\{2|v'| \leqq |v|, |v| \geqq 1\}$. First sum over v_i so that (77) is given by

$$\langle [w^2 w_1] \{\phi^{ij} * v_i \partial_{\beta_1}[\mu^{1/2} \partial^{\alpha_1} g_1]\} \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2, \partial^{\alpha}_{\beta} g_3 \rangle.$$

We again expand $\phi^{ij}(v-v')$ as in (85). By (50), we have the estimates (86) and (87). Plugging (86) and (87) into (85), we thus conclude that the v integrand in (77) can be majorized by

$$C \sum \int [1+|v'|]^2 \mu^{1/4}(v') |\partial^{\alpha_1}_{\bar\beta} g_1(v')| dv'$$
$$\times w^2 \{\sigma^{ij} \partial_i \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2\}^{1/2} \{\sigma^{ij} v_i v_j |\partial^{\alpha}_{\beta} g_3|^2\}^{1/2}$$
$$\leqq C \sum |\partial^{\alpha_1}_{\bar\beta} g_1|_{\ell} \{w^2 \sigma^{ij} \partial_i \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2\}^{1/2} \{w^2 \sigma^{ij} v_i v_j |\partial^{\alpha}_{\beta} g_3|^2\}^{1/2}.$$

By further integrating over v, this is bounded by the right-hand side of (71). Thus, the proof of (71) is complete.

The proof of (72) now follows from the Sobolev embedding $H^2(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3)$ and (71). Without loss of generality, assume $|\alpha_1| + |\bar\beta| \leqq N/2$ in (71). Then

$$\left(\sup_x |\partial^{\alpha_1}_{\bar\beta} g_1(x)|_{\ell}\right) |\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2|_{\sigma,\ell,\vartheta} + \left(\sup_x |\partial^{\alpha_1}_{\bar\beta} g_1(x)|_{\sigma,\ell}\right) |\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2|_{\ell,\vartheta}$$
$$\leqq \left(\sum \|\partial^{\alpha'}_{\beta'} g_1\|_{\ell}\right) |\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2|_{\sigma,\ell,\vartheta} + \left(\sum \|\partial^{\alpha'}_{\beta'} g_1\|_{\sigma,\ell}\right) |\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2|_{\ell,\vartheta},$$

where the summation is over $|\alpha'| + |\beta'| \leqq \frac{N}{2} + 2 \leqq N$. We deduce (72) by integrating (71) over $\mathbb{T}^3$ and using this computation.

4. Energy Estimate and Global Existence

In this section we will prove the energy estimate which is a crucial step in constructing global solutions. By now, it is standard to prove local existence of small solutions using the estimates either in Section 2 for the Boltzmann case or Section 3 for the Landau case:

Theorem 3. *For any sufficiently small $M^* > 0$, $T^* > 0$ with $T^* \leqq \frac{M^*}{2}$ and*

$$\frac{1}{2}\sum_{|\alpha|+|\beta|\leqq N} ||\partial^\alpha_\beta f_0||^2_{|\beta|-l,\vartheta} \leqq \frac{M^*}{2},$$

there is an unique classical solution $f(t,x,v)$ to (5) *in either the Boltzmann or the Landau case in* $[0,T^*) \times \mathbb{T}^3 \times \mathbb{R}^3$ *such that*

$$\sup_{0\leqq t\leqq T^*}\left\{\frac{1}{2}\sum_{|\alpha|+|\beta|\leqq N} ||\partial^\alpha_\beta f||^2_{|\beta|-l,\vartheta}(t) + \int_0^t \mathcal{D}_{l,\vartheta}(f)(s)ds\right\} \leqq M^*,$$

and $\frac{1}{2}\sum_{|\alpha|+|\beta|\leqq N} ||\partial^\alpha_\beta f||^2_{|\beta|-l,\vartheta}(t) + \int_0^t \mathcal{D}_{l,\vartheta}(f)(s)ds$ is continuous over $[0,T^*)$.

Next, we define some notation. For fixed $N \geqq 8$, $0 \leqq m \leqq N$ and $\vartheta, q, l \geqq 0$, a modified instant energy functional satisfies

$$\frac{1}{C}\mathcal{E}^m_{l,\vartheta}(g)(t) \leqq \sum_{|\beta|\leqq m, |\alpha|+|\beta|\leqq N} ||\partial^\alpha_\beta g(t)||^2_{|\beta|-l,\vartheta} \leqq C\mathcal{E}^m_{l,\vartheta}(g)(t). \tag{89}$$

Similarly the modified dissipation rate is given by

$$\mathcal{D}^m_{l,\vartheta}(g)(t) \equiv \sum_{|\beta|\leqq m, |\alpha|+|\beta|\leqq N} ||\partial^\alpha_\beta g(t)||^2_{\mathbf{D},|\beta|-l,\vartheta}. \tag{90}$$

Note that, $\mathcal{E}^N_{l,\vartheta}(g)(t) = \mathcal{E}_{l,\vartheta}(g)(t)$ and $\mathcal{D}^N_{l,\vartheta}(g)(t) = \mathcal{D}_{l,\vartheta}(g)(t)$. And as before, we will write $\mathcal{E}^m_{l,0}(g)(t) = \mathcal{E}^m_l(g)(t)$ and $\mathcal{D}^m_{l,0}(g)(t) = \mathcal{D}^m_l(g)(t)$. Now we are ready to state a result from equation (4.5) in [14] using this new notation:

Lemma 11. *Let $f(t,x,v)$ be a classical solution to* (5) *satisfying* (15) *in either the Boltzmann or the Landau case. In the Boltzmann case assume $\tau \leqq \gamma$ but in the Landau case assume $\tau \leqq -1$ in* (10). *For any $l \geqq 0$, there exists M_l, $\delta_l = \delta_l(M_l) > 0$ such that if*

$$\frac{1}{2}\sum_{|\alpha|+|\beta|\leqq N} ||\partial^\alpha_\beta f||^2_{|\beta|-l}(t) \leqq M_l, \tag{91}$$

then for any $0 \leqq m \leqq N$ we have an instant energy functional such that

$$\frac{d}{dt}\mathcal{E}^m_l(f)(t) + \mathcal{D}^m_l(f)(t) \leqq C\sqrt{\mathcal{E}_l(f)(t)}\mathcal{D}_l(f)(t). \tag{92}$$

We will bootstrap this energy estimate without an exponential weight ($\vartheta = 0$) to Corollary 1 in the Boltzmann case and Lemma 9 on the Landau case to obtain the following general energy estimate.

Lemma 12. *Fix $N \geqq 8$, $0 < \vartheta \leqq 2$, $q > 0$ and $l \geqq 0$. If $\vartheta = 2$ let $0 < q < 1$. In the Boltzmann case assume $\tau \leqq \gamma$ but in the Landau case assume $\tau \leqq -1$ in* (10). *Let $f(t,x,v)$ be a classical solution to* (5) *satisfying* (15) *and* (91) *in either the Boltzmann or the Landau case. For any given $0 \leqq m \leqq N$ there is a modified instant energy functional such that*

$$\frac{d}{dt}\mathcal{E}^m_{l,\vartheta}(f)(t) + \mathcal{D}^m_{l,\vartheta}(f)(t) \leqq C\mathcal{E}^{1/2}_{l,\vartheta}(f)(t)\mathcal{D}_{l,\vartheta}(f)(t). \tag{93}$$

Proof. We use an induction over m, the order of the $v-$derivatives. For $m = 0$, by taking the pure ∂^α derivatives of (5) we obtain

$$\{\partial_t + v \cdot \nabla_x\}\,\partial^\alpha f + L\{\partial^\alpha f\} = \sum_{\alpha_1 \leqq \alpha} C^{\alpha_1}_\alpha \Gamma(\partial^{\alpha_1} f, \partial^{\alpha-\alpha_1} f). \tag{94}$$

Multiply $w^2(-l,\vartheta)\partial^\alpha f$ with (94), integrate over $\mathbb{T}^3 \times \mathbb{R}^3$ and sum over $|\alpha| \leqq N$ to deduce the following for some constant $C > 0$,

$$\sum_{|\alpha|\leqq N}\left\{\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f(t)\|^2_{-l,\vartheta} + \Big(w^2(-l,\vartheta)L\{\partial^\alpha f(t)\}, \partial^\alpha f(t)\Big)\right\}$$
$$\leqq C\mathcal{E}^{1/2}_{l,\vartheta}(f)(t)\mathcal{D}_{l,\vartheta}(f)(t). \tag{95}$$

We have used Lemma 3 in the Boltzmann case and (72) in the Landau case to bound the right-hand side of (94). Notice that Lemma 2 (in the Boltzmann case) implies

$$\begin{aligned}\Big(w^2(-l,\vartheta)L\{\partial^\alpha f(t)\}, \partial^\alpha f(t)\Big) &= \|\partial^\alpha f(t)\|^2_{\boldsymbol{\nu},-l,\vartheta} \\ &\quad - \Big(w^2(-l,\vartheta)K\{\partial^\alpha f(t)\}, \partial^\alpha f(t)\Big) \\ &\geqq \frac{1}{2}\|\partial^\alpha f(t)\|^2_{\boldsymbol{\nu},-l,\vartheta} - C\|\partial^\alpha f(t)\|^2_{\boldsymbol{\nu},-l},\end{aligned}$$

where $C > 0$ is a large constant. In the Landau case, Lemma 9 gives

$$\begin{aligned}\Big(w^2(-l,\vartheta)L\{\partial^\alpha f(t)\}, \partial^\alpha f(t)\Big) &\geqq \delta_q^2\|\partial^\alpha f(t)\|^2_{\boldsymbol{\sigma},-l,\vartheta} - C\|\bar{\chi}_C\partial^\alpha f(t)\|^2_{-l} \\ &\geqq \delta_q^2\|\partial^\alpha f(t)\|^2_{\boldsymbol{\sigma},-l,\vartheta} - C\|\partial^\alpha f(t)\|^2_{\boldsymbol{\sigma},-l}.\end{aligned}$$

Without loss of generality say $0 < \delta_q < \frac{1}{2}$. Then in either case, plugging these into (95) we have

$$\sum_{|\alpha|\leqq N}\left\{\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f(t)\|^2_{-l,\vartheta} + \delta_q^2\|\partial^\alpha f(t)\|^2_{\boldsymbol{D},-l,\vartheta} - C\|\partial^\alpha f(t)\|^2_{\boldsymbol{D},-l}\right\}$$
$$\leqq C\sqrt{\mathcal{E}_{l,\vartheta}(f)(t)}\mathcal{D}_{l,\vartheta}(f)(t).$$

Add this inequality to (92) with $m = 0$, possibly multiplied by a large constant, to obtain (93) with $m = 0$.

Now assume the Lemma is valid for some fixed $m > 0$. For $|\beta| = m+1$, taking ∂^α_β of (5) we obtain

$$\begin{aligned}&\{\partial_t + v\cdot\nabla_x\}\,\partial^\alpha_\beta f + \partial_\beta\{L\partial^\alpha f\} \\ &= -\sum_{|\beta_1|=1} C^{\beta_1}_\beta \partial_{\beta_1} v\cdot\nabla_x \partial^\alpha_{\beta-\beta_1} f + \sum C^{\alpha_1}_\alpha \partial_\beta \Gamma(\partial^{\alpha_1} f, \partial^{\alpha-\alpha_1} f).\end{aligned} \tag{96}$$

We take the inner product of (96) with $w^2(|\beta| - l, \vartheta)\partial^\alpha_\beta f$ over $\mathbb{T}^3\times\mathbb{R}^3$. The first inner product on the left is equal to $\frac{1}{2}\frac{d}{dt}||\partial^\alpha_\beta f(t)||^2_{|\beta|-l,\vartheta}$. From either Corollary 1 in the Boltzmann case or Lemma 9 in the Landau case, we deduce that the inner product of $\partial_\beta\{L\partial^\alpha f\}$ is bounded from below as

$$\sum_{|\beta|=m+1}\left(w^2\partial_\beta\{L\partial^\alpha f\}, \partial^\alpha_\beta f\right) \geqq \delta_q^2 \sum_{|\beta|=m+1} ||\partial^\alpha_\beta f||^2_{\boldsymbol{D},|\beta|-l,\vartheta} - C\sum_{|\bar\beta|\leqq m} ||\partial^\alpha_{\bar\beta} f||^2_{\boldsymbol{D},|\bar\beta|-l,\vartheta}.$$

Since $|\beta_1| = 1$, by (10) and (14), as in [10] the streaming term on the right-hand side of (96) is bounded by

$$\begin{aligned}(w^2(|\beta| - l,\vartheta)\{\partial_{\beta_1} v_j\}\partial_{x_j}\partial^\alpha_{\beta-\beta_1} f, \partial^\alpha_\beta f) &\leqq \int w^2(|\beta| - l,\vartheta)|\partial_{x_j}\partial^\alpha_{\beta-\beta_1} f\,\partial^\alpha_\beta f|\,dx\,dv \\ &\leqq \left\| w(|\beta| + 1/2 - l,\vartheta)\partial^\alpha_\beta f\right\| \left\| w(1/2 + \{|\beta| - 1\} - l,\vartheta)\partial_{x_j}\partial^\alpha_{\beta-\beta_1} f\right\| \\ &\leqq \eta\left\|\partial^\alpha_\beta f\right\|^2_{\boldsymbol{D},|\beta|-l,\vartheta} + C_\eta\left\|\partial_{x_j}\partial^\alpha_{\beta-\beta_1} f\right\|^2_{\boldsymbol{D},|\beta-\beta_1|-l,\vartheta}.\end{aligned}$$

Further, by Lemma 3 in the Boltzmann case and Lemma 10 in the Landau case, the inner product involving Γ on the right-hand side of (96) is $\leqq C\mathcal{E}^{1/2}_{l,\vartheta}(f)(t)\mathcal{D}_{l,\vartheta}(f)(t)$.

Collect terms and sum over $|\beta| = m+1$, $|\alpha| + |\beta| \leqq N$ to obtain

$$\begin{aligned}&\sum_{|\beta|=m+1,|\alpha|+|\beta|\leqq N}\left\{\frac{1}{2}\frac{d}{dt}||\partial^\alpha_\beta f(t)||^2_{|\beta|-l,\vartheta} + \left(\delta_q^2 - W\eta\right)\left\|\partial^\alpha_\beta f\right\|^2_{\boldsymbol{D},|\beta|-l,\vartheta}\right\} \\ &\leqq \sum_{|\beta|=m+1,|\alpha|+|\beta|\leqq N} C\left(\sum_{|\beta_1|=1}\left\|\partial_{x_j}\partial^\alpha_{\beta-\beta_1} f\right\|^2_{\boldsymbol{D},|\beta-\beta_1|-l,\vartheta} + \sum_{|\bar\beta|\leqq m}||\partial^\alpha_{\bar\beta} f||^2_{\boldsymbol{D},|\beta|-l,\vartheta}\right) \\ &\qquad + CZ_{m+1}\mathcal{E}^{1/2}_{l,\vartheta}(f)(t)\mathcal{D}_{l,\vartheta}(f)(t).\end{aligned}$$

Here $Z_{m+1} = \sum_{|\beta|=m+1,|\alpha|+|\beta|\leqq N} 1$ and $W = \sum_{|\beta_1|=1} C^{\beta_1}_\beta$. Choosing $\eta > 0$ such that $\delta_q^2 - W\eta = \frac{\delta_q^2}{2} > 0$ we get

$$\begin{aligned}&\sum_{|\beta|=m+1,|\alpha|+|\beta|\leqq N}\left\{\frac{1}{2}\frac{d}{dt}||\partial^\alpha_\beta f(t)||^2_{|\beta|-l,\vartheta} + \frac{\delta_q^2}{2}\left\|\partial^\alpha_\beta f\right\|^2_{\boldsymbol{D},|\beta|-l,\vartheta}\right\} \\ &\leqq \tilde{C}\sum_{|\beta|\leqq m,|\alpha|+|\beta|\leqq N}||\partial^\alpha_\beta f||^2_{\boldsymbol{D},|\beta|-l,\vartheta} + CZ_{m+1}\sqrt{\mathcal{E}_{l,\vartheta}(f)(t)}\mathcal{D}_{l,\vartheta}(f)(t).\end{aligned}$$

Choose $\bar{A}_{m+1}$ such that $\bar{A}_{m+1} - \tilde{C} \geqq 1$. Now multiply (93) for $|\beta| \leqq m$ by $\bar{A}_{m+1}$ and add it to the display above to obtain (93) for $|\beta| \leqq m+1$. We thus conclude the energy estimate. $\square$

With Lemma 12, we can prove existence of global in time solutions with an exponential weight using exactly the same argument as in the last section of [11].

5. Proof of Exponential Decay

In this section we prove exponential decay using the differential inequality (16) and the uniform bound (17) with $\vartheta > 0$. The main difficulty in establishing decay from (16) is rooted in the fact that the dissipation $\mathcal{D}_{l,\vartheta}(f)(t)$ is in general weaker than the instant energy $\mathcal{E}_{l,\vartheta}(f)(t)$. As in the work of CAFLISCH [1], the key point is to split $\mathcal{E}_l(f)(t)$ into a time dependent low velocity part

$$E = \{|v| \leqq \rho t^{p'}\},$$

and its complementary high velocity part $E^c = \{|v| > \rho t^{p'}\}$, where $p' > 0$ and $\rho > 0$ will be chosen at the end of the proof.

First consider the Boltzmann case. Let $\mathcal{E}_l^{low}(f)(t)$ be the instant energy restricted to E. Then from (18), for $t > 0$, we have

$$\mathcal{D}_l(f)(t) \geqq C\rho^\gamma t^{\gamma p'} \mathcal{E}_l^{low}(f)(t). \tag{97}$$

Plugging this into the the differential inequality (16) we obtain

$$\frac{d}{dt}\mathcal{E}_l(f)(t) + C\rho^\gamma t^{\gamma p'} \mathcal{E}_l^{low}(f)(t) \leqq 0.$$

Letting $\mathcal{E}_l^{high}(f)(t) = \mathcal{E}_l(f)(t) - \mathcal{E}_l^{low}(f)(t)$ we have

$$\frac{d}{dt}\mathcal{E}_l(f)(t) + C\rho^\gamma t^{\gamma p'} \mathcal{E}_l(f)(t) \leqq C\rho^\gamma t^{\gamma p'} \mathcal{E}_l^{high}(f)(t).$$

Define $\lambda = C\rho^\gamma / p$, where for now $p = \gamma p' + 1$ and $p' > 0$ is arbitrary. Then

$$\frac{d}{dt}\mathcal{E}_l(f)(t) + \lambda p t^{p-1} \mathcal{E}_l(f)(t) \leqq \lambda p t^{p-1} \mathcal{E}_l^{high}(f)(t).$$

Equivalently

$$\frac{d}{dt}\left(e^{\lambda t^p}\mathcal{E}_l(f)(t)\right) \leqq \lambda p t^{p-1} e^{\lambda t^p} \mathcal{E}_l^{high}(f)(t).$$

The integrated form is

$$\mathcal{E}_l(f)(t) \leqq e^{-\lambda t^p}\mathcal{E}_l(f_0) + \lambda p e^{-\lambda t^p} \int_0^t s^{p-1} e^{\lambda s^p} \mathcal{E}_l^{high}(f)(s) ds.$$

Above $p > 0$ or equivalently $\gamma p' > -1$ is assumed to guarantee the integral on the right-hand side is finite. Since $\mathcal{E}_l^{high}(f)(s)$ is on $E^c = \{|v| > \rho s^{p'}\}$

$$\mathcal{E}_l^{high}(f)(s) = \mathcal{E}_{l,0}^{high}(f)(s) \leqq C e^{-\frac{q}{2}\rho s^{\vartheta p'}} \mathcal{E}_{l,\vartheta}^{high}(f)(s).$$

In the last display we have used the region and

$$1 \leqq \exp\left(\frac{q}{2}(1+|v|^2)^{\frac{\vartheta}{2}}\right) e^{-\frac{q}{2}|v|^{\vartheta}} \leqq \exp\left(\frac{q}{2}(1+|v|^2)^{\frac{\vartheta}{2}}\right) e^{-\frac{q}{2}\rho s^{\vartheta p'}}.$$

Hence (17) implies

$$\mathcal{E}_l(f)(t) \leqq e^{-\lambda t^p}\left(\mathcal{E}_{l,0}(f_0) + \lambda p \mathcal{E}_{l,\vartheta}(f_0)\int_0^t s^{p-1} e^{\lambda s^p - \frac{q}{2}\rho s^{\vartheta p'}} ds\right).$$

The biggest exponent p that we can allow with this splitting is $p = \vartheta p'$; since also $p = \gamma p' + 1$ we have $p' = \frac{1}{\vartheta - \gamma}$ so that

$$p = \frac{\gamma}{\vartheta - \gamma} + 1 = \frac{\vartheta}{\vartheta - \gamma}.$$

Further choose $\rho > 0$ large enough so that $\lambda = C\rho^{\gamma}/p < \frac{q}{2}\rho$ $(\gamma < 0)$ and hence

$$\int_0^{\infty} s^{p-1} e^{\lambda s^p - \frac{q}{2}\rho s^p} ds < +\infty.$$

This completes the proof of decay in the Boltzmann case.

For the proof of decay in the Landau case, instead of (97), we use Lemma 5 to see that

$$\mathcal{D}_l(f)(t) \geqq C\rho^{2+\gamma} t^{(2+\gamma)p'} \mathcal{E}_l^{low}(f)(t).$$

The rest of the proof is exactly the same but we find, in this case, that $p = \frac{\vartheta}{\vartheta - (2+\gamma)}$.
Q.E.D.

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