Supplementary exam problems for Chapter 9

1. When the line segment from $(0,0)$ to $(1,1)$ is rotated around the x-axis, it forms a cone.

   (a) What is the surface area of this cone?

   (b) Imagine that the line segment is made of an elastic material, anchored at $(0,0)$ and $(1,1)$, and someone alters the shape by pushing downward (toward the x-axis) on the middle. The resulting curve is then rotated about the x-axis. Is the new surface area certainly less than the area of the cone, or certainly greater, or can we not tell from the information given?

   (c) Suppse instead that the curve is altered by pushing upward (away from the x-axis), then rotated about the x-axis as before. Again, compare the area of the new surface of rotation to the area of the cone – certainly greater, certainly less, or can't tell?

2. A coke can (just the empty can) is a cylinder with height 12 cm, radius 3 cm, and weight 2 gms.

   (a) At what height is the center of mass of the empty can?

   (b) The can in now filled to height h with coke (density $1\,\text{gm}/\text{cm}^3$. How high is the center of mass of the liquid inside the can?

   (c) How high is the center of mass of the partially filled can (liquid plus can)?

   (d) The partially filled can will be the most stable when the center of mass is the lowest. How high should you fill the can for maximum stability?

3. A wire of length 28 cm is bent into some kind of curve in the half-plane $x > 0$. It is a monotone curve (moving along the curve so that the x-coordinate increases, also increases the y-coordinate). Rotating this about the y-axis produces a surface with area $300\pi\,\text{cm}^2$. If instead, it is rotated around the line $x = -1\,\text{cm}$, what will be the resulting surface area?

4. A radioactive material has a decay rate of $\lambda\,\text{sec}^{-1}$, meaning that each particle has a random lifetime, whose probability density is given by $\lambda e^{-\lambda t}$ for $t > 0$ measured in seconds. (When it reaches the end of its lifetime, it changes into a different material.)

   (a) What is the average lifetime of a particle?

   (b) If we start with a mass M of the material, what is the average mass of this material over the first 10 seconds?

5. A blood vessel has radius R and the flow at radius $r < R$ is given by $c(R - r)^2$ (compare with problem #22 in Section 6.5).

   (a) If a point is chosen at random on the cross-section (a circle of radius R), what is the probability density of the resulting distance r from the cetral axis of the blood vessel?

   (b) What is the average (over this probability distribution) of the flow rate $c(R - r)^2$?

   (c) Does this average, or the one in 6.5 # 22 or neither, compute the average flow rate through the vessel (total flow through the cross-section divided by area of the cross-section)?

6. Suppose X is continuous random variable with density f. What is the density of the random variable X/λ, where λ is a positive real number?

7. If a and b are positive integers, the beta density with parameters a and b is defined by $c_{a,b}\, x^{a-1}(1-x)^{b-1}$, where $c_{a,b}$ is a normalizing constant.

   (a) Compute $c_{3,2}$ (see Problem # 6 from Section 9.5 of the book).

   (b) Use integration by parts to relate $c_{a,b}$ to $c_{a+1,b-1}$.

   (c) Use this relation to compute $c_{a,b}$ by relating it to $c_{a+b-1,1}$.

   (d) Compute the mean of the beta distributions.

8. Let W be the sum of the x-coordinate and y-coordinate of a point chosen uniformly at random from the square $0 \le x, y \le 1$. What is the density of this random variable?

9. Choose a point p uniformly at random in the unit interval. Now let Y be a point chosen uniformly at random from the interval $(0, p)$. The density of Y is one of the following functions on the unit interval. Guess which one is right by ruling out all the others.

$$
\begin{aligned}
f(x) &= 1 \\
f(x) &= 12x^2(1-x) \\
f(x) &= 1-x \\
f(x) &= \ln(1/x) \\
f(x) &= \frac{1}{x}
\end{aligned}
$$

10. Suppose that X is a random variable and the probability of X being less than a is equal to $(1/\pi)\arccos(-a)$, for $-1 \leq a \leq 1$. What is the probability density function for X?

11. Two positive random variables, X and Y, are chosen independently with density f. Let $F(x) = \int_0^x f(t)dt$ denote the integral of f. Let W be the maximum of X and Y. Prove that the density of W is $2fF$. [Hint: the probability that W is less than or equal to a is $F(a)^2$.]

12. There is a continuous function f whose average value on every interval $[0, t]$ is equal to $\sin(t)/t$. What is f?