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WELCOME

to MATH 104:

Calculus I

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Welcome to the Course

1. **Penn Math 104 – Calculus I**
2. **Topics: quick review of high school calculus, methods and applications of integration, infinite series and applications, some functions of several variables.**
3. **College-level pace and workload:**

   Moves very fast - twelve sessions to do everything!
   Demanding workload, but help is available!
   YOU ARE ADULTS - how much do you need to practice each topic?

   Emphasis on applications - what is this stuff good for?
4. **Opportunities to interact with instructor, TA, and other students**

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Outline for Week 1

(a)Review of functions and graphs

(b)Review of limits

(c)Review of derivatives - idea of velocity, tangent and normal lines to curves

(d)Review of related rates and max/min problems

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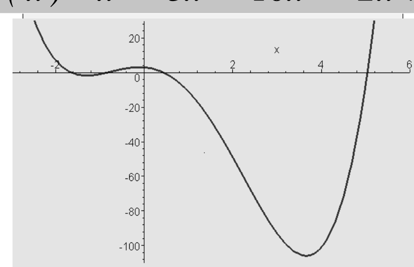
Functions and Graphs

The idea of a function and of the graph of a function should be very familiar

$$f(x) = x^4 - 3x^3 - 10x^2 - 2x + 3$$



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Questions for discussion...

1. **Describe the graph of the function f(x) (use calculus vocabulary as appropriate).**
2. **The graph intersects the y-axis at one point. What is it (how do you find it)?**
3. **How do you know there are no other points where the graph intersects the y-axis?**
4. **The graph intersects the x-axis at four points. What are they (how do you find them)?**
5. **How do you know there are no other points where the graph intersects the x-axis?**
6. **The graph has a low point around x=4, y=-100. What is it exactly? How do you find it?**
7. **Where might this function come from?**

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Kinds of functions that should be familiar:

Linear, quadratic

Polynomials, quotients of polynomials

Powers and roots

Exponential, logarithmic

Trigonometric functions (sine, cosine, tangent, secant, cotangent, cosecant)

Hyperbolic functions (sinh, cosh, tanh, sech, coth, csch)

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Quick Question

The domain of the function

$$f(x) = \frac{\sqrt{1-x}}{x^2 - 2x}$$ **is...**

A. All x except x=0, x=2
B. All x $\leq$ 1 except x=0.
C. All x $\geq$ 1 except x=2.
D. All x $\leq$ 1.
E. All x $\geq$ 1.

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Quick Question

Which of the following has a graph that is symmetric with respect to the y-axis?

A. y= $\dfrac{x-1}{x}$

B. y= $2x^4 - 1$

C. y= $x^3 - 2x$

D. y= $x^5 - 2x^2$

E. y= $\dfrac{x}{x^3+3}$

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Quick Question

The period of the function

$$f(x) = \sin\left(\frac{3\pi x}{5}\right)$$ is...

A. 3
B. 3/5
C. 10/3
D. 6/5
E. 5

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Quick Question

If $\log_a\left(5^a\right) = \dfrac{a}{3}$, then a=

A. 5
B. 15
C. 25
D. 125
E. None of these

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Limits

Basic facts about limits

The concept of limit underlies all of calculus.
Derivatives, integrals and series are all different kinds of limits.

Limits are one way that mathematicians deal with the infinite.

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First things first...

First some notation and a few basic facts.
Let *f* be a function, and let *a* and L be fixed numbers.

Then $\lim_{x \to a} \mathrm{f}(x) = L$ **is read**

"the limit of f(x) as x approaches a is L"

You probably have an intuitive idea of what this means.
And we can do examples:

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For many functions...

...and many values of a , it is true that

$$\lim_{x\to a} f(x) = f(a)$$

And it is usually apparent when this is not true.

"Interesting" things happen when $f(a)$ is not well-defined, or there is something "singular" about f at a .

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Definition of Limit

So it is pretty clear what we mean by

$$\lim_{x\to a} f(x) = L$$

But what is the formal mathematical definition?

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Properties of real numbers

One of the reasons that limits are so difficult to define is that a limit, if it exists, is a real number. And it is hard to define precisely what is meant by the system of real numbers.

Besides algebraic and order properties (which also pertain to the system of rational numbers), the real numbers have a *continuity* property.

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Least upper bound property

If a set of real numbers has an upper bound, then it has a *least* upper bound.

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Important example

The set of real numbers x such that $x^2 < 2$. The corresponding set of rational numbers has no least upper bound. But the set of reals has the number $\sqrt{2}$

In an Advanced Calculus course, you learn how to start from this property and construct the system of real numbers, and how the definition of limit works from here.

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Official definition

$\lim_{x\to a} f(x) = L$ means that for any $\varepsilon > 0$,

no matter how small, you can find a $\delta > 0$

such that if x is within δ of a, i.e., if $|x\text{-}a| < \delta$,

then $|f(x)\text{-}L| < \varepsilon$

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For example….

$$\lim_{x\to 5} x^2 = 25$$

because if $\varepsilon < 1$ and we choose $\delta < \frac{1}{11}\varepsilon$

Then for all x such that $|x-5| < \delta$ we have

$5-\delta < x < 5+\delta$ and so

$$25-10\delta+\delta^2 < x^2 < 25+10\delta+\delta^2$$

which implies

$$|x^2-25| < 10\delta+\delta^2 < 10\frac{\varepsilon}{11}+\frac{\varepsilon^2}{121} < \varepsilon$$

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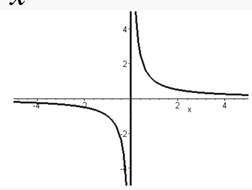
Top ten famous limits:

1. $\displaystyle\lim_{x\to 0+}\frac{1}{x} = \infty$ $\qquad$ $\displaystyle\lim_{x\to 0-}\frac{1}{x} = -\infty$



2. $\displaystyle\lim_{x\to\infty}\frac{1}{x} = 0$

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3. (A) If $0 < x < 1$ then $\displaystyle\lim_{n\to\infty} x^n = 0$

(B) If $x > 1$, then $\displaystyle\lim_{n\to\infty} x^n = \infty$

4. $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$ **and** $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x} = 0$

5. $\displaystyle\lim_{x\to -\infty} \mathrm{e}^x = 0$ **and** $\displaystyle\lim_{x\to\infty} \mathrm{e}^x = \infty$

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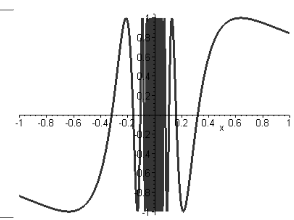
6. For *<u>any</u>* value of *n*, $\displaystyle\lim_{x\to\infty}\frac{x^n}{e^x} = 0$

and for any *positive* value of *n*, $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x^n} = 0$

7. $\displaystyle\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$

does not exist!



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8. $\displaystyle\lim_{x\to 0+} x\ln(x) = 0$

9. $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

10. If *f* is differentiable at *a*, then

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = f'(a)$$

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Basic properties of limits

I. Arithmetic of limits:

If both $\displaystyle\lim_{x\to a} f(x)$ **and** $\displaystyle\lim_{x\to a} g(x)$ **exist, then**

$$\lim_{x\to a} f(x)\pm g(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x)\lim_{x\to a} g(x)$$

and if $\displaystyle\lim_{x\to a} g(x) \neq 0$ **, then** $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

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II. Two-sided and one-sided limits:

$\lim_{x \to a} f(x) = L$ if and only if

BOTH $\lim_{x \to a+} f(x) = L$ and $\lim_{x \to a-} f(x) = L$

III. Monotonicity:

If $f(x) \le g(x)$ for all x near a,

then $\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$

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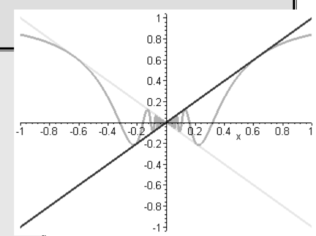
IV. Squeeze theorem:

If $f(x) \le g(x) \le h(x)$ for all x near a, and if $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$, then $\lim_{x \to a} g(x)$ exists and is equal to the common value of the other two limits.

$-|x| \le x \sin x \le |x|$

$\lim_{x \to 0} x \sin x = 0$



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Let's work through a few:

$$\lim_{x \to 2} \frac{x+5}{x+2} \qquad \lim_{x \to -2} \frac{x+5}{x+2}$$

$$\lim_{x \to -2} \frac{x^2-4}{x+2}$$

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Now you try this one...

$$\lim_{t \to 0} \frac{\sqrt{2-t} - \sqrt{2}}{t} =$$

A. 0	**E.** -1
B. ∞	**F.** $\sqrt{2}$
C. -1/2	**G.** -2
D. $\frac{1}{2\sqrt{2}}$	**H.** $-\frac{1}{2\sqrt{2}}$

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Continuity

A function f is *continuous at x = a* if it is true that $\lim_{x \to a} f(x) = f(a)$

(The existence of both the limit and of *f*(*a*) is implicit here).

Functions that are continuous at every point of an interval are called "*continuous on the interval*".

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Intermediate value theorem

The most important property of continuous functions is the "common sense" Intermediate Value Theorem:

Suppose f is continuous on the interval $[a,b]$, and $f(a) = m$, and $f(b) = M$, with $m < M$. Then for any number p between m and M, there is a solution in $[a,b]$ of the equation $f(x) = p$.

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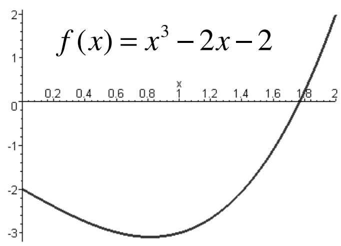
Application of the intermediate-value theorem



Since $f(0)$=-2 and $f(2)$=+2, there must be a root of $f(x)$=0 in between x=0 and x=2. A naive way to look for it is the "*bisection method*" -- try the number halfway between the two closest places you know of where f has opposite signs.

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$$f(x) = x^3 - 2x - 2$$

We know that $f(0)$ = -2 and $f(2)$ = 2, so there is a root in between. Choose the halfway point, x = 1.

Since f(1) = -3 < 0, we now know (of course, we already knew from the graph) that there is a root between 1 and 2. So try halfway between again:

$$f(1.5) = -1.625$$

So the root is between 1.5 and 2. Try 1.75:

$$f(1.75) = -.140625$$

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$$f(x) = x^3 - 2x - 2$$

We had $f(1.75) < 0$ and $f(2) > 0$. So the root is between 1.75 and 2. Try the average, x = 1.875

$$f(1.875) = .841796875$$

f is positive here, so the root is between 1.75 and 1.875. Try their average (x=1.8125):

$$f(1.8125) = .329345703$$

So the root is between 1.75 and 1.8125. One more:

$$f(1.78125) = .089141846$$

So now we know the root is between 1.75 and 1.8125.

You could write a computer program to continue this to any desired accuracy.

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Derivatives

Let's discuss it:

1. What, in a few words, is the derivative of a function?

2. What are some things you learn about the graph of a function from its derivative?

3. What are some applications of the derivative?

4. What is a differential? What does dy = f '(x) dx mean?

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Derivatives (continued)

Derivatives give a comparison between the rates of change of two variables:

When x changes by so much, then y changes by so much.

Derivatives are like "exchange rates".

6/03/02	1 US Dollar = 1.0650 Euro 1 Euro = 0.9390 US Dollar (USD)
6/04/02	1 US Dollar = 1.0611 Euro 1 Euro = 0.9424 US Dollar (USD)

Definition of derivative:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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Common derivative formulas:

$$\frac{d}{dx}\left(x^p\right) = px^{p-1}$$

$$\frac{d}{dx}\left(e^x\right) = e^x$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)\frac{dg}{dx} + \frac{df}{dx}g(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{\left(g(x)\right)^2}$$

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))g'(x)$$

Let's do some examples…..

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Derivative question #1

Find f '(1) if $f(x) = \sqrt[5]{x} + \dfrac{1}{x^{9/5}}$

A. 1/5
B. 2/5
C. -8/5
D. -2/5
E. -1/5
F. 4/5
G. 8/5
H. -4/5

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Derivative question #2

Find the equation of a line tangent to $y = \dfrac{8}{\sqrt{4+3x}}$ at the point (4,2).

A. 6x+y=26
B. 4x+2y=20
C. 3x-4y=4
D. 7x+18y=64
E. 5x+21y=62
F. 4x+15y=46
G. 3x+16y=44
H. 2x-y=6

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Derivative question #3

Calculate $\dfrac{d^2 f}{dx^2}$ if $f(x) = \dfrac{e^x}{x}$

A. $e^x(x+4)\big/x^4$
B. $e^x(x^2-1)\big/x^4$
C. $e^x(x^2+x)\big/x^4$
D. $e^x(x^2+3)\big/x^4$
E. $e^x(x-2)\big/x^3$
F. $e^x(x^2+5x)\big/x^3$
G. $e^x(x^2-2x+2)\big/x^3$
H. $e^x(x^3-4x^2+3)\big/x^3$

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Derivative question #4

What is the largest interval on which the function $f(x) = \dfrac{x}{x^2+1}$ is concave upward?

A. (0,1)
B. (1,2)
C. (1, ∞)
D. (0, ∞)
E. (1, $\sqrt{3}$)
F. ($\sqrt{3}$, ∞)
G. ($\sqrt{2}$, ∞)
H. (1/2, ∞)

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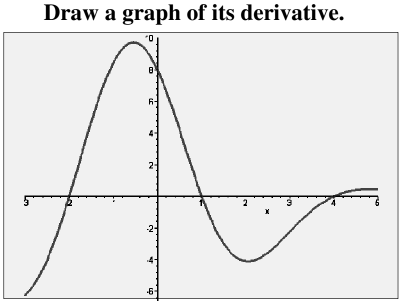
Discussion

Here is the graph of a function.

Draw a graph of its derivative.



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The meaning and uses of derivatives, in particular:

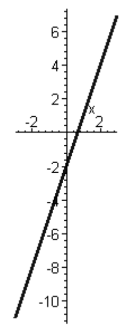
- (a) The idea of linear approximation
- (b) How second derivatives are related to quadratic functions
- (c) Together, these two ideas help to solve max/min problems

Basic functions --linear and quadratric.

- **The derivative and second derivative provide us with a way of comparing other functions with (and approximating them by) linear and quadratic functions.**
- **Before you can do that, though, you need to understand linear and quadratic functions.**

Let's review

- **Let's review: linear functions of one variable in the plane are determined by one point + slope (one number):**
- **y = 4 + 3(x-2)**



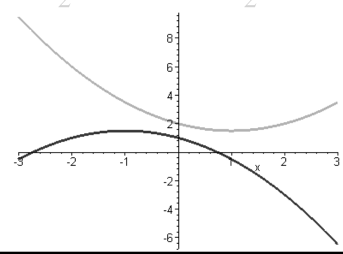
Linear functions

- **Linear functions occur in calculus as differential approximations to more complicated functions (or first-order Taylor polynomials):**
- **f(x) = f(a) + f '(a) (x-a) (approximately)**

Quadratic functions

- **Quadratic functions have parabolas as their graphs:**

$$y=\frac{x^2}{2}-x+2,\ y=-\frac{x^2}{2}-x+1$$



Quadratic functions

- **Quadratic functions occur as second-order Taylor polynomials:**
- **f(x) = f(a) + f '(a)(x-a) + f ''(a)(x-a)^2/2! (approximately)**

They also help us tell...

- **… relative maximums from relative minimums -- if f '(a) =0 the quadratic approximation reduces to**
- **f(x) = f(a) + f ''(a)(x-a)^2/2! and the sign of f ''(a) tells us whether x=a is a relative max (f ''(a)<0) or a relative min (f ''(a)>0).**

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Position, velocity, and acceleration:

You know that if y = f(t) represents the position of an object moving along a line, the v = f '(t) is its velocity, and a = f ''(t) is its acceleration.

Example: For falling objects, y = $y_0 + v_0 t - 16t^2$ **is the height of the object at time t, where** y_0 **is the initial height (at time t=0), and** v_0 **is its initial velocity.**

Also, by way of review, recall that to find the maximum and minimum values of a function on any interval, we should look at three kinds of points:

1. **The *critical points* of the function. These are the points where the derivative of the function is equal to zero.**
2. **The places where the derivative of the function fails to exist (sometimes these are called critical points,too).**
3. **The endpoints of the interval. If the interval is unbounded, this means paying attention to**

$$\lim_{x\to\infty} f(x) \text{ and/or } \lim_{x\to(-\infty)} f(x).$$

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Related Rates

Recall how related rates work. This is one of the big ideas that makes calculus important:

If you know how z changes when y changes (dz/dy) and how y changes when x changes (dy/dx), then you know how z changes when x changes:

$$\frac{dz}{dx} = \frac{dz}{dy}\,\frac{dy}{dx}$$

Remember the idea of implicit differentiation: The derivative of f(y) with respect to x is f '(y)$\frac{dy}{dx}$

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More on related rates

The idea is that "differentiating both sides of an equation with respect to x" [or any other variable] is a legal (and useful!) operation.

This is best done by using examples...

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Related Rates Greatest Hits

A light is at the top of a 16-ft pole. A boy 5 ft tall walks away from the pole at a rate of 4 ft/sec. At what rate is the tip of his shadow moving when he is 18 ft from the pole? At what rate is the length of his shadow increasing?

A man on a dock is pulling in a boat by means of a rope attached to the bow of the boat 1 ft above the water level and passing through a simple pulley located on the dock 8 ft above water level. If he pulls in the rope at a rate of 2 ft/sec, how fast is the boat approaching the dock when the bow of the boat is 25 ft from a point on the water directly below the pulley?

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Greatest Hits...

A weather balloon is rising vertically at a rate of 2 ft/sec. An observer is situated 100 yds. from a point on the ground directly below the balloon. At what rate is the distance between the balloon and the observer changing when the altitude of the balloon is 500 ft?

The ends of a water trough 8 ft long are equilateral triangles whose sides are 2 ft long. If water is being pumped into the trough at a rate of 5 cu ft/min, find the rate at which the water level is rising when the depth is 8 in.

Gas is escaping from a spherical balloon at a rate of 10 cu ft/hr. At what rate is the radius changing when the volume is 400 cu ft?

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Integrals

Start with dx -- this means "a little bit of x" or

"*a little change in x*"

If we add up a whole bunch of little changes in x, we get the "*total change of x*" --

A tautology question: If you add up all the *changes in x* as x changes from 2 to 7, what do you get?

A. 0

B. 2

C. 5

D. 7

E. cannot be determined

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We write this in integral notation as: $\int_2^7 1\,dx = 5$

If y = f(x), then we write dy = f '(x) dx.

To add up all the "*little changes in y*" as x changes from 2 to 7, we should write $\int_2^7 f'(x)dx$ or $\int_2^7 \frac{df}{dx}dx$

... and the answer should be the total change in y as x changes from 2 to 7, in other words

$$\int_2^7 \frac{df}{dx}dx = f(7) - f(2)$$

This is the content of the fundamental theorem of calculus!

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The fundamental theorem of calculus…

gives the connection between derivatives and integrals. It says you can calculate

$$\int_a^b g(x)dx$$

precisely if you can find a function whose derivative is g(x). And the result is the difference between the value of the "anti-derivative" function evaluated at b minus the same function evaluated at a.

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Basic antiderivative formulas:

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ except for n= -1

$\int \frac{1}{x}dx = \ln(x) + C$

$\int \cos(x)\,dx = \sin(x) + C$

$\int e^x dx = e^x + C$

$\int \sin(x)\,dx = -\cos(x) + C$

$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C$

$\int \frac{1}{1+x^2}dx = \arctan(x) + C$

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A quick example

Find the value of $\int_0^1 (1+x)^2 dx$

A. 7/3	E. 2
B. 0	F. 1/3
C. 1	G. 4/3
D. 5/3	H. 2/3

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Fundamental Theorem Workout

Let $f(x) = \int_x^{x^2} t^2 dt$

Find the value of f '(1) -- the *derivative of f at 1*.

A. 3	E. 5
B. 8	F. 2
C. 4	G. 6
D. 0	H. 1

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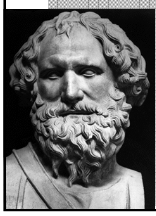
Integrals and Areas

A problem that was around long before the invention of calculus is to find the area of a general plane region (with curved sides).



And a method of solution that goes all the way back to Archimedes is to divide the region up into lots of little regions, so that you can find the area of almost all of the little regions, and so that the total area of the ones you can't measure is very small.

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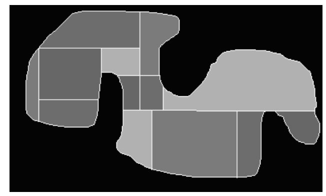
By Newton's time, people realized that it would be sufficient to handle regions that had three straight sides and one curved side (or two or one straight side -- the important thing is that all but one side is straight). Essentially all regions can be divided up into such regions.



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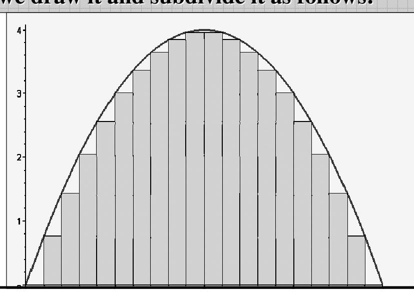
These all-but-one-side-straight regions look like areas under the graphs of functions. And there is a standard strategy for calculating (at least approximately) such areas. For instance, to calculate the area between the graph of $y = 4x - x^2$ and the x axis, we draw it and subdivide it as follows:



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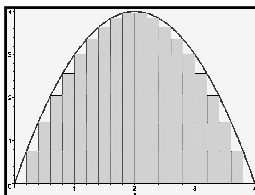

Since the green pieces are all rectangles, their areas are easy to calculate. The blue parts under the curve are relatively small, so if we add up the areas of the rectangles, we won't be far from the area under the curve. For the record, the total area of all the green rectangles is: $\frac{246}{25}$

whereas the actual area under the curve is:

$$\int_0^4 4x - x^2 \, dx = \frac{32}{3}$$

Also for the record, 246/25 = 9.84 while 32/3 is about 10.6667.

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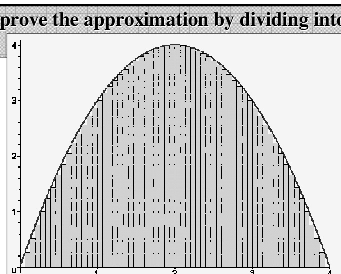
We can improve the approximation by dividing into more rectangles:



Now there are 60 boxes instead of 20, and their total area is: $\frac{7018}{675}$ which is about 10.397. Getting better. We can in fact take the limit as the number of rectangles goes to infinity, which will give the same value as the integral. *This was Newton's and Leibniz's great discovery -- derivatives and integrals are related and they are related to the area problem.*

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Limits of Riemann sums

A kind of limit that comes up occasionally is an integral described as the limit of a Riemann sum. One way to recognize these is that they are generally

expressed as , where the

“something” depends on *n* as well as on *i*.

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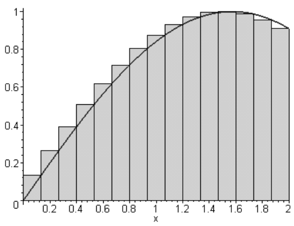
Green graph

Again, recall that one way to look at integrals is as areas under graphs, and we approximate these areas as sums of areas of rectangles.

This is a picture of the"right endpoint" approximation to the integral of a function.



If we are approximating the integral of $f(x)$ over the interval from a to b using n rectangles, then each rectangle has width $\frac{b-a}{n}$. The right side of the ith rectangle is at $x = a + \frac{i(b-a)}{n}$, and so the area of the ith rectangle is $f\left(a + \frac{i(b-a)}{n}\right)\frac{b-a}{n}$. The sum of the areas of the rectangles is thus $\sum_{i=1}^{n} \frac{(b-a)f(a+\frac{i(b-a)}{n})}{n}$, and the limit of this sum as n approaches infinity is the integral $\int_a^b f(x)dx$. An example will help

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Example... What is $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4}$?

First, we need a 1/n for our (b-a)/n. So we can rewrite the expression under the summation sign as $(\frac{1}{n})(\frac{i}{n})^3$. Now we need to figure out a and b. For (b-a)/n = 1/n, we need b-a = 1. And i/n appears in the other factor, we should choose a = 0. If f(x) = x^3, then we'll have $\frac{(b-a)f\left(a+\frac{i(b-a)}{n}\right)}{n} = \frac{1}{n}(\frac{i}{n})^3$. Thus the limit of the sum is equal to $\int_0^1 x^3 dx = \frac{1}{4}$.

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Position, velocity, and acceleration:

Since velocity is the derivative of position and acceleration is the derivative of velocity,

Velocity is the integral of acceleration, and position is the integral of velocity.

(Of course, you must know *starting values of position and/or velocity* to determine the constant of integration.)

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Example...

An object moves in a force field so that its acceleration at time t is a(t) = t^2-t+12 (meters per second squared). Assuming the object is moving at a speed of 5 meters per second at time t=0, determine how far it travels in the first 10 seconds.

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Solution...

First we determine the velocity, by integrating the acceleration. Because v(0) = 5, we can write the velocity v(t) as 5 + a *definite* integral, as follows:

$$v(t) = 5 + \int_0^t a(\tau)\,d\tau = 5 + \int_0^t \tau^2 - \tau + 12\,d\tau = 5 + \frac{t^3}{3} - \frac{t^2}{2} + 12t$$

The distance the object moves in the first 10 seconds is the total change in position. In other words, it is the integral of dx as t goes from 0 to 10. But dx = v(t) dt. So we can write:

(distance traveled between t=0 and t=10) = $\int_0^{10} v(t)\,dt$

= $\int_0^{10} 5 + \frac{t^3}{3} - \frac{t^2}{2} + 12t\,dt$ **= 3950/3 = 1316.666... meters .**

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Methods of integration

Before we get too involved with applications of the integral, we have to make sure we're good at calculating antiderivatives. There are four basic tricks that you have to learn (and hundreds of *ad hoc* ones that only work in special situations):

1. **Integration by substitution (chain rule in reverse)**
2. **Trigonometric substitutions (using trig identities to your advantage)**
3. **Partial fractions (an algebraic trick that is good for more than doing integrals)**
4. **Integration by parts (the product rule in reverse)**

We'll do #1 this week, and the others later. LOTS of practice is needed to master these!

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Substitution

In some ways, substitution is the most important technique, because *every* integral can be worked this way (at least in theory).

The idea is to remember the chain rule: If G is a function of u and u is a function of x, then the derivative of G with respect to x is:

$$\frac{dG}{dx} = G'(u)\,u'(x)$$

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For instance...

e^{x^2} **could be considered as e^u where $u = x^2$.**

To differentiate e^{x^2} **then, we use that the derivative of e^u is e^u :**

$$\frac{d}{dx}e^{x^2} = \frac{d}{du}\left(e^u\right)\frac{d}{dx}\left(x^2\right) = e^u(2x) = 2xe^{x^2}$$

Now we'll turn this around...

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To do an integral problem...

For a problem like $\int x^3 e^{x^4}\,dx$

we suspect that the x^4 should be considered as u

and then $x^3\,dx$ is equal to $du/4$.

And so:

$$\int x^3 e^{x^4}\,dx = \int e^u \frac{du}{4} = \frac{1}{4}\int e^u\,du = \frac{1}{4}e^u + C = \frac{1}{4}e^{x^4} + C$$

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In general...

In substitution, you

1. **Separate the integrand into factors**
2. **Figure out which factor is the most complicated**
3. **Ask whether the *other* factors are the derivative of some (compositional) part of the complicated one.**

This provides the clue as to what to set u equal to.

Here's another one:

$\int \frac{x}{(2x^2+5)^3}\,dx$ -- **the complicated factor is clearly the denominator (partly by virtue of being in the denominator!)**

and the rest (x dx) is a constant times the differential of x^2 -- but it's a good idea to try and make u substitute for as much of the complicated factor as possible.

And if you think about it, x dx is a constant times the differential of $2x^2+5$! So we let $u = 2x^2+5$, then $du = 4x\,dx$, in other words $x\,dx = du / 4$. So we can substitute:

$$\int \frac{x}{(2x^2+5)^3}\,dx = \tfrac{1}{4}\int \frac{1}{u^3}\,du = \tfrac{1}{4}\frac{u^{-2}}{-2} + C = -\frac{1}{8(2x^2+5)^2} + C$$

Now you try a couple...

$$\int_0^{\sqrt{\pi/2}} x\cos x^2\,dx =$$

A) 0

B) 1/2

C) 1

D) $\pi/2$

E) $\sqrt{\pi}$

Find $\int_0^{\pi/4} \sec^2 x \sin(\tan x)\,dx$

A) $\pi/2$

B) $1-\pi/4$

C) sin 1

D) 1 - cos 1

E) $\pi/2$ - sin 1

F) $\pi/4$ + cos 1

G) $1 + 3\pi/4$

H) 1 + tan 1